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# Matrix STRETCHING for Linear Equations\*

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## Abstract

Stretching is a new sparse matrix method that makes matrices sparser by making them larger. Stretching has implications for computational complexity theory and applications in scientific and parallel computing. It changes matrix sparsity patterns to render linear equations more easily solved by parallel and sparse techniques. Some stretchings increase matrix condition numbers only moderately, and thus solve linear equations stably. For example, these stretchings solve arrow equations with accuracy and expense preferable to other solution methods.

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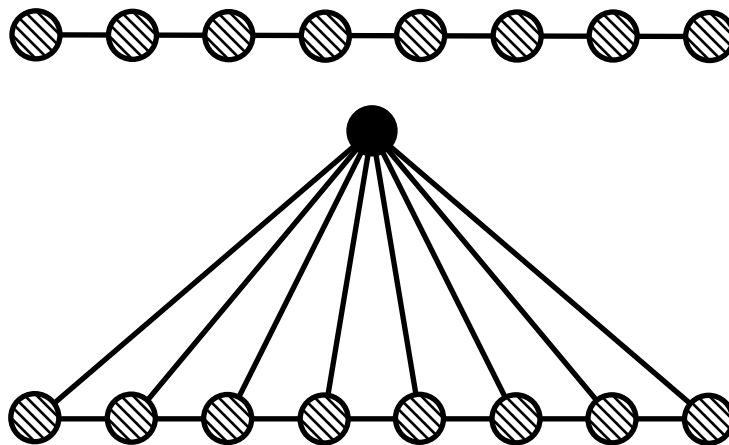


# 1. Introduction

Many matrices of computational interest contain mostly zeroes and so are called *sparse*. Yet sparse algorithms exploit both the quantity of zeroes and the placement of nonzeros, so in a practical sense sparse matrices are those with a few nonzeros in the right place. *Stretching* is a new sparse matrix method that increases sparsity by rearranging the nonzeros into larger matrices. Some systems of linear equations can be solved more easily by stretching them first. Stretching thereby addresses two fundamental issues in scientific computing.

**Question 1.** *What are the limits of easy parallelism?*

Computations with uniform data dependencies lend themselves to parallel execution, but small changes to regular dependency structures inhibit parallelism. Figure 1 shows an uniform structure with a disastrous perturbation. The irregularity may represent a globally synchronized task or globally shared data. Both are troublesome to parallel machines of various kinds. Question 1 asks whether these irregularities necessarily block parallelism.

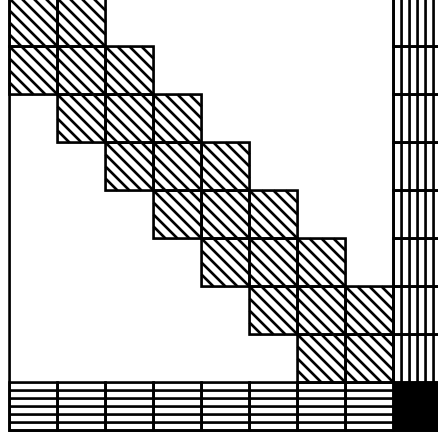


**Figure 1.** *Dependency structures affording easy and uneasy parallelism.*

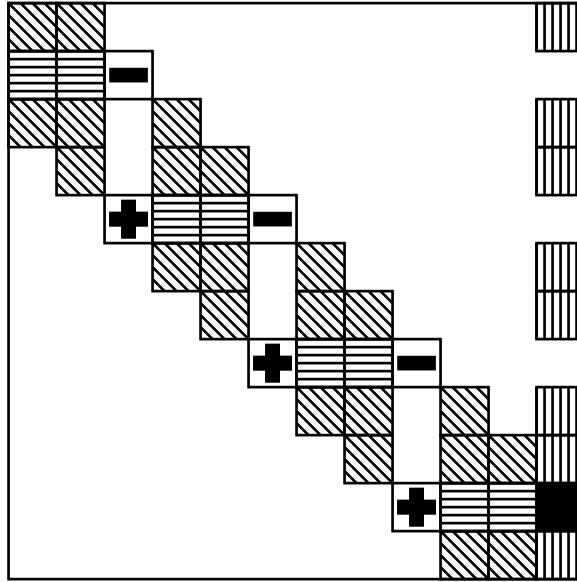
Figure 1's dependency structures occur in scientific computing as *occupancy graphs* for sparse matrices. This terminology is new but the concept is well known. For a system of linear equations written in matrix notation,  $Ax = y$ , the matrix diagonal positions become the graph's vertices, and if the row of one vertex has a nonzero entry in the column of another, then an edge connects the two vertices.<sup>1</sup> Figure 2 displays a matrix whose occupancy graph is the distorted one of Figure 1. The dense column represents a variable that appears in every equation, the dense row represents an equation that includes all the variables, and both are common in problems from linear programming and differential equations. Dense rows and

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<sup>1</sup> An edge connects vertices  $j$  and  $k$  when a nonzero occupies matrix entry  $(j, k)$ . Parter [12] originated the study of Gaussian elimination using these graphs [5, p. 4] but didn't name them. The sparse matrix literature now prefers "the graph of the matrix" or the *matrix graph*, but still doesn't award the concept a formal definition or a separate place in the index. See also [6]. Conversely, the matrix whose entry  $(j, k)$  is nonzero when an edge connects vertices  $j$  and  $k$  is well known in combinatorial mathematics as the *adjacency matrix* of a graph. Both concepts extend to directed graphs, and may include loop edges  $(j, j)$ .



**Figure 2.** Matrix whose occupancy graph is the irregular one in Figure 1. This matrix is visually sparse but functionally dense.



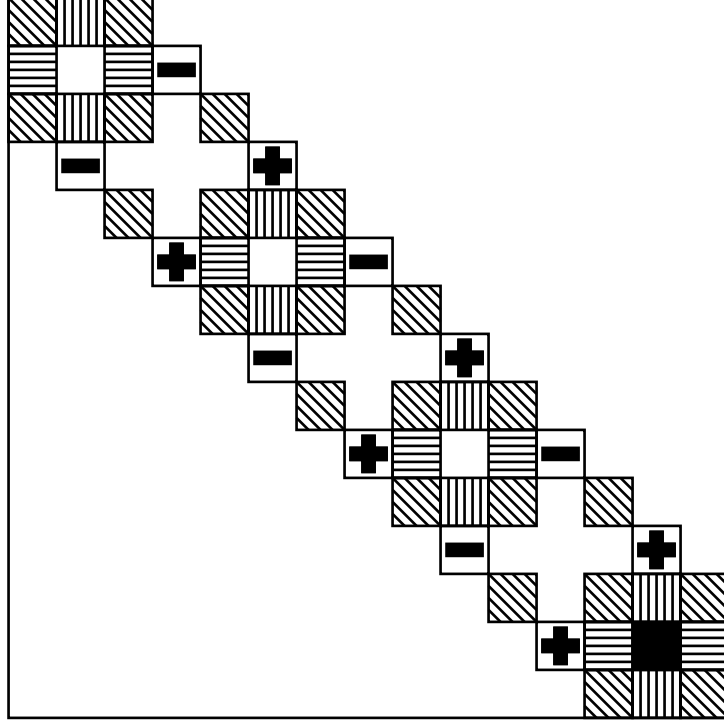
**Figure 3.** Sparser form of the matrix in Figure 2 obtained by row stretching. The stretched rows sum to the original dense row.

columns entail slow global communication on computers with massive parallelism and distributed memories. They complicate load balancing on computers with limited parallelism and shared memories.

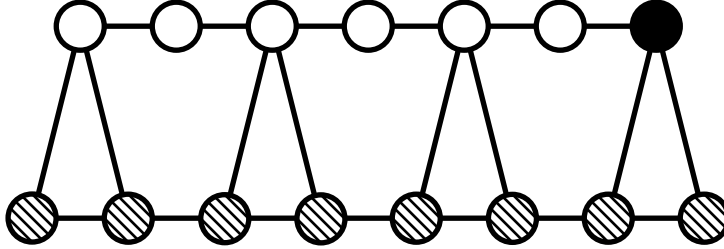
Stretching removes dense rows and columns that frustrate parallel processing. Figures 3 and 4 exhibit stretched versions of Figure 2's matrix, and Figure 5 shows the altered occupancy graph. These particular stretchings move entries of dense rows and columns into new, sparser rows and columns. They glue the scattered pieces together by introducing some new nonzeros. Compared to the original matrices, the stretched matrices are larger and have the same nonzeros in different places. Whence the name *stretching*.

**Question 2.** What is the price of accuracy?

Computational complexity theory usually treats a single algorithm and so over-



**Figure 4.** Sparsest form of the matrix in Figure 2 obtained by row and column stretching. The stretched rows and columns sum to the original dense row and column, respectively.



**Figure 5.** Occupancy graph for the matrix in Figure 4. Stretching makes the matrix in Figure 2 sparse, and makes the irregular dependency graph in Figure 1 uniform.

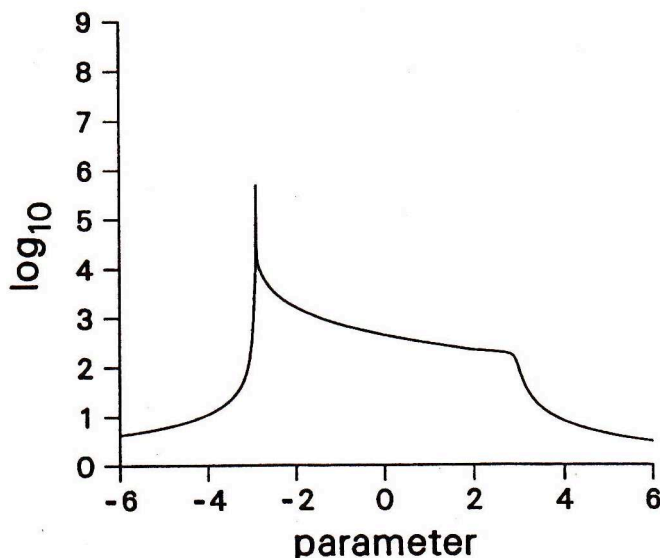
looks a central concern in scientific computing. More complex algorithms may be needed to maintain accuracy when a problem's data changes.<sup>2</sup>

The complexity of finding accurate solutions can be a strongly discontinuous function of the problem. This is illustrated by linear equations with the irregular dependencies of Figures 1 and 2 whose coefficient matrices vary with a parameter. Figure 6 shows the matrices are well-conditioned so it is feasible to ask for accurate solutions. Figures 7 and 8 show the accuracy and complexity vary greatly. Some parameter values demand much more complex solution algorithms.

The increased complexity stems from the reordering algorithms that stabilize

<sup>2</sup> The serial time complexity of a calculation is the number of operations it performs, the space complexity is the number of memory cells it touches. For systems of linear equations solved by matrix factorization, space complexity is roughly the nonzero population of the factors.

## 2-Norm Condition Number



**Figure 6.** 2-norm condition numbers for parameterized matrices of order 51 with sparsity patterns like the matrix in Figure 2. Appendix 2 and Section 1 explain the calculations.

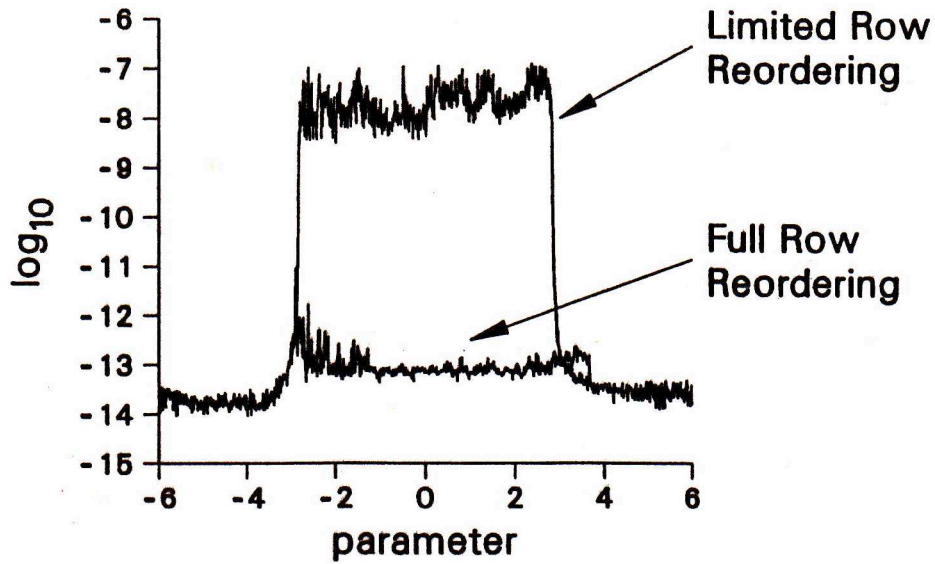
matrix computations. The complexity in Figure 8 jumps when reordering is needed to maintain uniformly low errors in Figure 7, as follows. If reordering selects a dense row to participate at an early stage of the factorization, it engenders more of the same, and increases the likelihood that additional dense rows will be selected, and created. So many zeroes may be lost in this way that the factors become completely dense and the complexity becomes very high.

Stretching removes dense rows and columns that make reordering expensive. Stretched matrices have only sparse rows and columns and therefore have fewer or no reorderings that entail many nonzeros. Although stretched matrices are larger, they are likely factored more easily. Figures 9 and 10 display the accuracy and complexity when the matrices of Figure 6 stretch in the manner of Figure 3. The accuracy matches Figure 7's best; the complexity almost matches Figure 8's lowest. Stretching achieves high accuracy and low complexity.

Many scientific calculations implicitly avoid matrices with inconvenient sparsity patterns. The final section of this paper describes a precedent that inspires matrix stretching: analytic transformations that ease numerical solution of some differential equations. This paper is the first step toward making stretching a purely algebraic—and therefore a broadly applicable—tool of scientific computation.

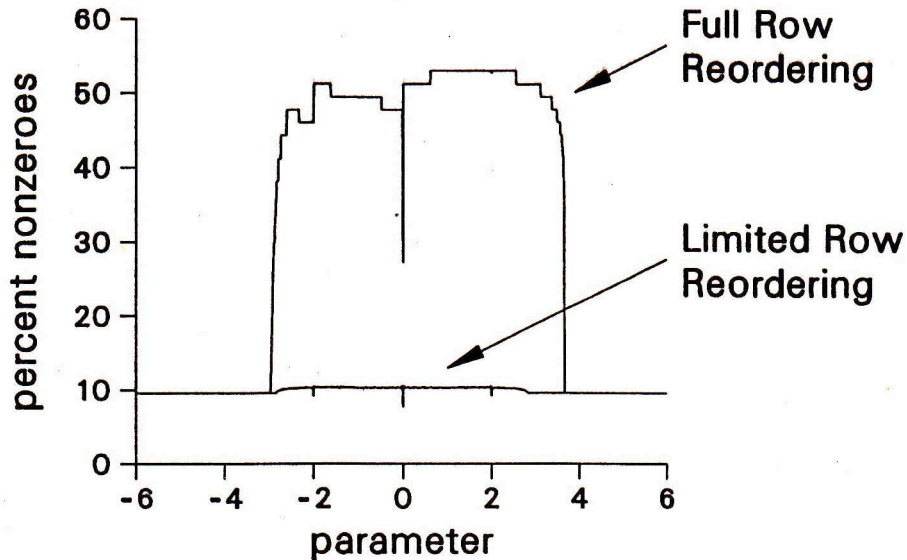
These are the paper's major results. First, stretching is recognized as a sparse matrix method with implications beyond numerical linear algebra and with potentially widespread applications. It does not appear in the sparse matrix literature, but it has been used indirectly to prepare some differential equations for numerical solution. Second, some stretchings are shown to increase matrix condition numbers moderately. The proof of this is different from others in linear algebra and may have independent interest. Third, the a priori error bounds for solving linear equations are proved to increase only slightly with stretching. Fourth, stretching's reliability

## 2-Norm Relative Error



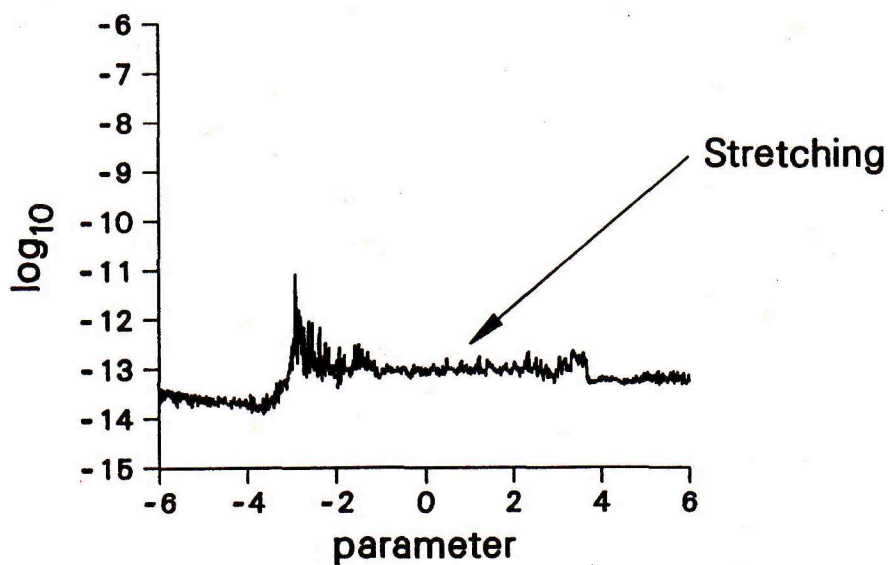
**Figure 7.** Maximum 2-norm relative errors for equations  $Ax = y$ , with 20 different  $y$ 's and the parameterized matrices  $A$  of Figure 6, solved by triangular factorization. The lower curve allows full row reordering. The upper curve restricts row reordering to the tridiagonal band. Appendix 2 and Section 1 explain the calculations.

## Complexity



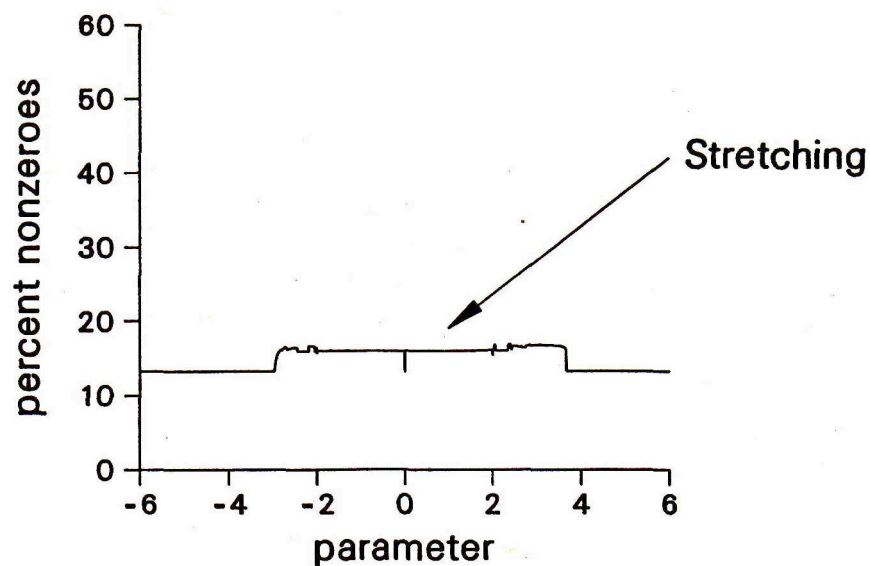
**Figure 8.** Percent of non-zeroes in the triangular factors of the matrices of Figure 6. The upper curve allows full row reordering. The lower curve restricts row reordering to the tridiagonal band. Appendix 2 and Section 1 explain the calculations.

## 2-Norm Relative Error



**Figure 9.** 2-norm relative errors for the equations of Figure 7 solved by triangular factorization with full row reordering after stretching in the manner of Figure 3. Appendix 2 and Section 1 explain the calculations.

## Complexity



**Figure 10.** Percent of non-zeroes in the triangular factors of the stretched matrices of Figure 9. The percentages are relative to the size of the unstretched matrices. Appendix 2 and Section 1 explain the calculations.

and economy are demonstrated by the special class of arrow equations for which stretching is found preferable to other solution methods.

This paper is organized as follows. Section 2 presents a general framework for constructing stretchings that solve linear equations. The stretchings that implicitly accompany some differential equations follow naturally in Section 3, where they are christened *simple row and column stretchings*. Section 4 shows these stretchings stably solve linear equations when some parameters are properly chosen. Application to arrow matrices is made in Section 5, and comparison with deflated block elimination is made in Section 6. Finally, Section 7 describes the differential transformations that inspired this work. Odd-numbered sections are specific and accessible; Section 2 is more general and introduces notation used throughout the paper; Sections 4 and 6 are more technical. Applications to parallel processing and randomly sparse matrices are not developed beyond the suggestions made in this Introduction. To improve readability, appendices contain proofs of theorems and descriptions of numerical experiments.

## 2. Stretching Equations

What is needed to begin is a *stretching* process that associates the matrix  $A$  with a larger matrix  $A^S$ .

$$A \rightarrow A^S$$

The reason for stretching is something about  $A$  makes  $Ax = y$  difficult to solve and something about  $A^S$  makes  $A^S z = y^S$  easier. Stretchings and *squeezings* are needed for vectors too. The overall solution process then consists of first stretching  $A \rightarrow A^S$  and  $y \rightarrow y^S$ , next solving  $A^S z = y^S$ , and finally squeezing  $z \rightarrow z_S = x$ .

$$\begin{array}{ccc} A^S & z & = y^S \\ \uparrow & \downarrow & \uparrow \\ A & z_S & = y \end{array}$$

The superscript  $^S$  indicates something dimensionally bigger than the matrix or vector underneath, the subscript  $_S$  indicates something smaller. In this notation the process of solving linear equations is simply

$$x = ((A^S)^{-1} y^S)_S.$$

Little is gained by greater formalism. Of interest rather are stretchings and squeezings that work. They can be anything at all provided the result is something useful like  $Ax = y$ .

Matrix stretchings with the ancillary vector operations needed to solve linear equations are difficult to find. The stretchings illustrated in Figures 2, 3 and 4 fit a common pattern which is of interest because it may aid the discovery of more. The pattern springs from a sequence of assumptions which might be altered to obtain different stretchings. The first assumption is (1) the stretched matrix be square and nonsingular if the original is. Alternate courses are possible, for example, stretching might produce under- or over-determined equations to be solved by least squares methods.

The next assumption is (2) the vector operations be linear and more or less independent of  $A$  and  $A^S$ .

$$\begin{aligned} y &\rightarrow y^S := Y^- y \\ z &\rightarrow z_S := {}^-X z \end{aligned}$$

Alternate courses might employ affine transformations. If the stretchings and squeezings solve  $Ax = y$  for all  $y$ , then linearity implies

$$x = ((A^S)^{-1}y^S)_S = {}^-X(A^S)^{-1}Y^-y$$

and makes the search for stretchings the search for *oversize* factorizations.

$$A^{-1} = {}^-X(A^S)^{-1}Y^-$$

There are many of these, but not many whose factors  ${}^-X$  and  $Y^-$  are independent of  $A$  and  $A^S$ . Lacking some independence the vector operations could degenerate to applying  $A^{-1}$  and stretching would gain nothing.

An acceptable situation has the factors depending at most on the sparsity patterns of  $A$  and  $A^S$ . Factorizations with restrictions of this kind are unlikely to be found even with explicit knowledge of  $A^{-1}$ . Theorem 1 provides a mechanism to overcome this difficulty by parameterizing the oversize factorizations of  $A^{-1}$ .

**Theorem 1.** *If  $A$  and  $A^S$  are nonsingular and*

*for some matrix  $Y$                       or                      for some matrix  $X$*

$$\begin{array}{l|l} {}^-X := A^{-1}YA^S & {}^-X := \text{any left inverse of } X \\ Y^- := \text{any right inverse of } Y & Y^- := A^SXA^{-1} \end{array}$$

*then  $A^{-1} = {}^-X(A^S)^{-1}Y^-$  (proof appears in Appendix 1).*

The notational symmetry,  $X$  and  $Y$ ,  ${}^-X$  and  $Y^-$ , is suggested by the Theorem's corollary.

**Corollary to Theorem 1.** *If in addition*

$$X := (A^S)^{-1}Y^-A \quad \Bigg| \quad Y := A{}^-X(A^S)^{-1}$$

*then  ${}^-XX = I$ ,  $YY^- = I$  and  $A = Y A^S X$  (proof appears in Appendix 1).*

The third, more restrictive assumption is (3)  ${}^-X$  and  $Y^-$  be built from one of the Theorem's two sets of formulas. Alternate courses might seek different expressions for  ${}^-X$  and  $Y^-$ , but the formulas in Theorem 1 allow considerable freedom. A likely stretching  $A \rightarrow A^S$  might have several matrices  $Y$  and  $X$  which yield factorizations for  $A^{-1}$ , produced by the formulas above, that are appropriate for solving linear equations. The sole criteria in choosing among them is the convenience of applying the  ${}^-X$  and  $Y^-$  actually used to solve equations.

Something concrete begins to appear if the parametric matrix  $Y$  or  $X$  alone participates in the Corollary's factorization of  $A$ , that is, if either

$$\begin{array}{l|l} A^S := [B \quad G]P^t & A^S := P^t \begin{bmatrix} B \\ G \end{bmatrix} \\ \text{and } YB = A & \text{and } BX = A \end{array}$$

in which  $P$  is a permutation matrix. The extra columns and rows, both denoted  $G$  for *glue*, can do more than make the stretched matrices square. When they lie in the null spaces of  $Y$  or  $X$ , then Theorem 1's  ${}^-X$  or  $Y^-$  depend only on  $P$ .

$$\begin{array}{l|l} {}^-X = A^{-1}YA^S = [I \quad 0]P^t & Y^- = A^SXA^{-1} = P^t \begin{bmatrix} I \\ 0 \end{bmatrix} \\ \text{provided } YG = 0 & \text{provided } GX = 0 \end{array}$$

The null space condition therefore makes  $\neg X$  or  $Y^-$  independent of  $A^{-1}$ , which is assumption (2). Moreover, if  $A^S$  is nonsingular then the extra columns or rows necessarily are linearly independent, but with the null space condition conversely, if the extra columns and rows are linearly independent then  $A^S$  is nonsingular, which is assumption (1). This leads to the fourth and final assumption, which makes the search for stretchings the search for one-sided factorizations of  $A$ . It is embodied in the following Definition. The subsequent Theorem 2 formalizes the preceding discussion and validates the use of Definition 1's *row and column stretchings* to solve linear equations.

**Definition 1, Row and Column Stretchings.** A row or column stretching  $A \rightarrow A^S$  of square matrices has

row stretching  $A^S := [B \ G] P^t$  with $YB = A$ and $YG = 0$  for some $Y$ of full rank	column stretching  $A^S := P^t \begin{bmatrix} B \\ G \end{bmatrix}$  with $BX = A$ and $GX = 0$  for some $X$ of full rank
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in which  $G$  has full rank and  $P$  is a permutation matrix, and chooses

$\neg X := [I \ 0] P^t$  $Y^- := \text{any right inverse of } Y$	$Y^- := P^t \begin{bmatrix} I \\ 0 \end{bmatrix}$  $\neg X := \text{any left inverse of } X.$
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**Theorem 2.** If  $A \rightarrow A^S$  is a row or column stretching and  $A$  is nonsingular, then  $A^S$  is nonsingular and  $A^{-1} = \neg X (A^S)^{-1} Y^-$  (proof appears in Appendix 1).

**Corollary to Theorem 2.** If  $A \rightarrow A^S$  is a row or column stretching of a nonsingular matrix  $A$ , if  $\neg X$  and  $Y^-$  are the auxiliary matrices in Definition 1, and if  $A^S z = y^S$  are the stretched equations corresponding to  $Ax = y$ , then not only  $\neg X z = x$  but also  $z = Xx$  (proof appears in Appendix 1).

In summary, finding stretchings to solve equations involves two tasks. One is to find better  $A^S$ , and assuming linear vector operations, the other is to find workable  $\neg X$  and  $Y^-$ . Row and column stretchings are valuable because they are a rich class of matrix stretchings for which acceptable  $\neg X$  and  $Y^-$  are readily available.

Row stretchings can be viewed as being built in three stage. The first,

$${}_n A_n \rightarrow {}_{n+m} B_n \quad \text{with} \quad {}_n Y_{n+m} B_n = {}_n A_n,$$

increases the row dimension in a way reversible by multiplication with some matrix  $Y$ . Whence the name *row stretching*. The new notation,  ${}_{n+m} B_n$ , indicates a matrix of  $n + m$  rows and  $n$  columns. The second stage,

$${}_{n+m} B_n \rightarrow [{}_{n+m} B_n \quad {}_{n+m} G_m] \quad \text{with} \quad {}_n Y_{n+m} G_m = 0,$$

adds new columns annihilated by  $Y$ . The third,

$$[B \ G] \rightarrow [B \ G] P^t = A^S,$$

scrambles the columns in a way reversible by a permutation matrix  $P$ .

Column stretchings are the transpose of row stretchings. Again there are three stages. The first,

$${}_n A_n \rightarrow {}_n B_{n+m} \quad \text{with} \quad {}_n B_{n+m} X_n = {}_n A_n,$$

increases only the column dimension in a way reversible by multiplication with some matrix  $X$ . Whence the name *column stretching*. The second stage,

$${}_n B_{n+m} \rightarrow \begin{bmatrix} {}_n B_{n+m} \\ {}_m G_{n+m} \end{bmatrix} \quad \text{with} \quad {}_m G_{n+m} X_n = 0,$$

adds new rows annihilated by  $X$ . The third,

$$\begin{bmatrix} B \\ G \end{bmatrix} \rightarrow P^t \begin{bmatrix} B \\ G \end{bmatrix} = A^S,$$

scrambles the rows in a way reversible by a permutation matrix  $P$ .

### 3. Simple Stretchings

The Introduction's stretchings receive a proper christening here. Section 7 describes their prior use in the numerical solution of ordinary differential equations, but now they are seen to be legitimate offspring of general algebraic methods, and are named *simple row and column stretchings*. The following derivation amounts to making specific choices for  $B$  and  $G$  in Definition 1.

A situation in which row stretching may be of use is that of a single dense row which inhibits row reordering during triangular factorization. This row represents a linear equation of the form

$$a_{j,1}x_1 + a_{j,2}x_2 + a_{j,3}x_3 + a_{j,4}x_4 + a_{j,5}x_5 + a_{j,6}x_6 = y_j$$

$$\Updownarrow$$

$$[a_{j,1} \quad a_{j,2} \quad a_{j,3} \quad a_{j,4} \quad a_{j,5} \quad a_{j,6}]$$

in which  $a_{j,k}$ ,  $x_k$  and  $y_j$  are entries of  $A$ ,  $x$  and  $y$  in  $Ax = y$ . Row stretching might be used to expand this row into something sparser.

$$\begin{bmatrix} a_{j,1} & a_{j,2} & & & & \\ & & a_{j,3} & a_{j,4} & & \\ & & & & a_{j,5} & a_{j,6} \end{bmatrix}$$

Section 4 shows this choice can reduce the computational complexity of triangular factorization. The first stage of row stretching,  $A \rightarrow B$ , simply replaces the  $j^{th}$  row of  $A$  by the three stretched rows above and optionally reorders the rows. This stage is undone by a transformation  $YB = A$  that copies the untouched rows and sums the three stretched ones. If row  $j$  is the last in  $A$  and if the stretched rows replace it at the bottom, then

$$Y = \begin{bmatrix} 1 & & & & & \\ & 1 & & & & \\ & & 1 & & & \\ & & & 1 & & \\ & & & & 1 & \\ & & & & & 1 & 1 & 1 \end{bmatrix}.$$

The second stage,  $B \rightarrow [B \ G]$ , produces a square matrix by appending new columns which, to make the stretched matrix nonsingular, must span the right null space of  $Y$ . A column vector in this null space has zeroes in the original rows of  $A$  and sums to 0 over the stretched rows. After the second stage the stretched rows could be

$$\begin{bmatrix} a_{j,1} & a_{j,2} & & & & -\sigma_1 \\ & & a_{j,3} & a_{j,4} & & +\sigma_1 & -\sigma_2 \\ & & & & a_{j,5} & a_{j,6} & +\sigma_2 \end{bmatrix}$$

for some nonzero  $\sigma_1$  and  $\sigma_2$ .

The third and final stage,  $[B \ G] \rightarrow A^S$ , reorders the columns. This is more than a cosmetic detail because column order affects the complexity of solving equations. The new columns could become the  $3^{rd}$  and  $6^{th}$ . Altogether  $A^S$  has the following stretched rows.

$$\begin{bmatrix} a_{j,1} & a_{j,2} & -\sigma_1 & & & & \\ & & +\sigma_1 & a_{j,3} & a_{j,4} & -\sigma_2 & \\ & & & & & +\sigma_2 & a_{j,5} & a_{j,6} \end{bmatrix}$$

$A^S$  can be used to solve  $Ax = y$  as follows. Step 1 forms  $y^S = Y^-y$  where  $Y^-$  is any right inverse for  $Y$ . This transformation copies entries of unstretched rows from  $y$  to  $y^S$  and places numbers that sum to  $y_j$  in the three stretched rows. Step 2 solves  $A^S z = y^S$ . Step 3 forms  $x = \neg X z$  where  $\neg X = [I \ 0]P^t$  and  $P$  is the permutation matrix that reorders the columns in stage 3. This means the old variables lie among the new in locations corresponding to the original columns of  $A$ . The net result is the original equation has been replaced by

$$\begin{aligned} a_{j,1}x_1 + a_{j,2}x_2 - \sigma_1 s_1 &= t_1 \\ + \sigma_1 s_1 + a_{j,3}x_3 + a_{j,4}x_4 - \sigma_2 s_2 &= t_2 \\ + \sigma_2 s_2 + a_{j,5}x_5 + a_{j,6}x_6 &= t_3 \end{aligned}$$

in which  $s_1$  and  $s_2$  are the new variables and any numbers that sum to  $y_j$  can appear on the right. Different choices give different values to the new variables, but of course the original variables remain unchanged.

Although column stretching is the transpose of row stretching, significant conceptual differences arise when solving equations. It is best to consider a separate example—taking care to avoid the page costs of displaying column vectors. A dense column represents a variable that occurs in several linear equations of the form

$$\begin{aligned} \dots + a_{1,k}x_k + \dots &= y_1 \\ \dots + a_{2,k}x_k + \dots &= y_2 \\ \dots + a_{3,k}x_k + \dots &= y_3 \\ \dots + a_{4,k}x_k + \dots &= y_4 \\ \dots + a_{5,k}x_k + \dots &= y_5 \\ \dots + a_{6,k}x_k + \dots &= y_6 \end{aligned} \iff \begin{bmatrix} a_{1,k} \\ a_{2,k} \\ a_{3,k} \\ a_{4,k} \\ a_{5,k} \\ a_{6,k} \end{bmatrix}$$

in which  $a_{j,k}$ ,  $x_k$  and  $y_j$  are entries of  $A$ ,

$x$  and  $y$  in  $Ax = y$ . Column stretching can be used to expand this one column to something sparser.

$$\begin{bmatrix} a_{1,k} \\ a_{2,k} \\ & a_{3,k} \\ & a_{4,k} \\ & & a_{5,k} \\ & & & a_{6,k} \end{bmatrix}$$

The first stage,  $A \rightarrow B$ , replaces the  $k^{th}$  column of  $A$  by the three stretched columns above and optionally reorders the columns. This stage is undone by a transformation  $BX = A$  that copies the untouched columns and sums the three stretched ones. If column  $k$  is the last in

$A$  and if the stretched columns replace it at the right side, then

$$X = \begin{bmatrix} 1 & & & & & \\ & 1 & & & & \\ & & 1 & & & \\ & & & 1 & & \\ & & & & 1 & \\ & & & & & 1 \\ & & & & & & 1 \\ & & & & & & & 1 \end{bmatrix}.$$

The second stage,

$$B \rightarrow \begin{bmatrix} B \\ G \end{bmatrix}$$

produces a square matrix by appending new rows that span the left null space of  $X$ . A row vector in this null space has zeroes in the original columns of  $A$  and sums to 0 over the stretched columns. After the second stage the stretched columns could be

$$\begin{bmatrix} a_{1,k} \\ a_{2,k} \\ & a_{3,k} \\ & a_{4,k} \\ & & a_{5,k} \\ & & a_{6,k} \\ -\sigma_1 & +\sigma_1 \\ & -\sigma_2 & +\sigma_2 \end{bmatrix}$$

for some nonzero  $\sigma_1$  and  $\sigma_2$ . The third stage reorders the rows. If the new rows become the 3<sup>rd</sup> and 6<sup>th</sup>, then  $A^S$  has stretched columns

$$\begin{bmatrix} a_{1,k} \\ a_{2,k} \\ -\sigma_1 & +\sigma_1 \\ & a_{3,k} \\ & a_{4,k} \\ & -\sigma_2 & +\sigma_2 \\ & & a_{5,k} \\ & & a_{6,k} \end{bmatrix}.$$

Once again  $A^S$  can be used to solve  $Ax = y$ , but the steps differ from the

row case in several details. Step 1 forms  $y^S = Y^-y$

$$Y^- = P^t \begin{bmatrix} I \\ 0 \end{bmatrix}$$

in which  $P$  is the permutation matrix that reorders the rows in stage 3. This transformation copies all entries of  $y$  into  $y^S$  and places zeroes in the new rows. Step 2 solves  $A^S z = y^S$ . Step 3 forms  $x = -Xz$  where  $-X$  can be any left inverse for  $X$ . Entries of  $z$  that correspond to original columns copy directly into  $x$ . That is, unstretched columns retain their original variables. Entries of  $z$  that correspond to stretched columns coalesce in a linear combination whose coefficients sum to 1. That is, the original variable  $x_k$  equals any linear combination, with coefficients summing to 1, of the new variables for the stretched columns. The net result is that the original equations have been replaced by

$$\begin{aligned} \dots + a_{1,k}s_1 & & + \dots & = y_1 \\ \dots + a_{2,k}s_1 & & + \dots & = y_2 \\ & -\sigma_1 s_1 + \sigma_1 s_2 & & = 0 \\ \dots + & a_{3,k}s_2 & + \dots & = y_3 \\ \dots + & a_{4,k}s_2 & + \dots & = y_4 \\ & -\sigma_2 s_2 + \sigma_2 s_3 & & = 0 \\ \dots + & & a_{5,k}s_3 + \dots & = y_5 \\ \dots + & & a_{6,k}s_3 + \dots & = y_6 \end{aligned}$$

in which  $s_1$ ,  $s_2$ , and  $s_3$  are the new variables. The equations make the new variables equal to  $x_k$  in principal, but machine computation makes them different in fact. Section 4 considers the effect of numerical error.

Differences between row and column stretching therefore occur in solving linear equations. Row stretching allows some freedom in choosing the right side of the stretched equations, but completely specifies how to recover the solution of the original equations. The reverse is true for column stretching. Column stretching completely specifies the right side, but allows some freedom in recovering the solution. The simple stretchings described above apply as well to blocks of rows and columns.

**Definition 2, Simple Row and Column Stretchings.** For a system of linear equations  $Ax = y$  whose coefficient matrix

$$A = \begin{bmatrix} A_1 \\ A_2 \end{bmatrix}$$

has a block of dense rows  $A_2$ , simple row stretching partitions the columns into  $m$  blocks

$$\begin{bmatrix} A_{1,1} & A_{1,2} & A_{1,3} & \dots & A_{1,m} \\ A_{2,1} & A_{2,2} & A_{2,3} & \dots & A_{2,m} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_m \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

and replaces the equations by

$$\begin{bmatrix} A_{1,1} & A_{1,2} & A_{1,3} & \dots & A_{1,m} & 0 & 0 & \dots & 0 \\ A_{2,1} & & & & & -D_1 & 0 & \dots & 0 \\ & A_{2,2} & & & & +D_1 & -D_2 & & \\ & & A_{2,3} & & & & +D_2 & \ddots & \\ & & & \ddots & & & & \ddots & -D_{m-1} \\ & & & & A_{2,m} & & & & +D_{m-1} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_m \\ s_1 \\ s_2 \\ \vdots \\ s_{m-1} \end{bmatrix} = \begin{bmatrix} y_1 \\ t_1 \\ t_2 \\ t_3 \\ \vdots \\ t_m \end{bmatrix}$$

in which  $D_1, D_2, \dots, D_{m-1}$  are nonsingular (presumably diagonal) matrices and  $y_2 = t_1 + t_2 + \dots + t_m$ . Alternatively, for a system of linear equations  $Ax = y$  whose coefficient matrix

$$A = [A_1 \quad A_2]$$

has a block of dense columns  $A_2$ , simple column stretching partitions the rows into  $m$  blocks

$$\begin{bmatrix} A_{1,1} & A_{1,2} \\ A_{2,1} & A_{2,2} \\ A_{3,1} & A_{3,2} \\ \vdots & \vdots \\ A_{m,1} & A_{m,2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_m \end{bmatrix}$$

and replaces the equations by

$$\begin{bmatrix} A_{1,1} & A_{1,2} & & & & & & & \\ A_{2,1} & & A_{2,2} & & & & & & \\ A_{3,1} & & & A_{3,2} & & & & & \\ \vdots & & & & \ddots & & & & \\ A_{m,1} & & & & & A_{m,2} & & & \\ 0 & -D_1 & +D_1 & & & & & & \\ 0 & & -D_2 & +D_2 & & & & & \\ \vdots & & & \ddots & \ddots & & & & \\ 0 & & & & -D_{m-1} & +D_{m-1} & & & \end{bmatrix} \begin{bmatrix} x_1 \\ s_1 \\ s_2 \\ s_3 \\ \vdots \\ s_m \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_m \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

in which  $D_1, D_2, \dots, D_{m-1}$  are nonsingular (presumably diagonal) matrices and  $x_2 = s_1 = s_2 = \dots = s_m$ . The matrices can be reordered both before and after the stretchings and may assume a final appearance quite different from the templates above.

**Theorem 3.** *A simple row stretching in the sense of Definition 2 is a row stretching in the sense of Definition 1, and similarly for column stretchings (proof appears in Appendix 1).*

The general row and column stretchings of Definition 1 are parameterized by auxiliary matrices  $X$  and  $Y$ , respectively. Ignoring reorderings, the simple row stretching of Definition 2 has

$$Y = \begin{bmatrix} I_1 & & & & \\ & I_2 & I_2 & \dots & I_2 \end{bmatrix}$$

in which  $I_1$  and  $I_2$  are identity matrices whose orders match the row orders of  $A_1$  and  $A_2$ . Simple column stretching has

$$X = \begin{bmatrix} I_1 & & & & \\ & I_2 & & & \\ & I_2 & & & \\ & \vdots & & & \\ & I_2 & & & \end{bmatrix}$$

where the orders of  $I_1$  and  $I_2$  match the column orders of  $A_1$  and  $A_2$ . Theorem 2 can be invoked with Theorem 3 and these  $X$  and  $Y$  to confirm the nonsingularity of the stretched matrices and the procedure for solving linear equations. Yet in this simple case these conclusions can be obtained more directly. Theorem 4 shows the stretched matrices are nonsingular.

**Theorem 4.** *If  $A \rightarrow A^S$  is a simple row or column stretching as in Definition 2, then*

$$\det A^S = \det A \prod_{j=1}^{m-1} \det D_j$$

*with perhaps a sign change when the rows and columns are reordered as the definition allows (proof appears in Appendix 1).*

## 4. Numerical Stability

Bounds on the rounding error for solving equations with and without stretching compare favorably because properly formed stretchings increase matrix condition numbers at worst moderately. This is the paper's major analytic result.

Analyses of stretching's errors must consider more than matrix condition numbers. The overall process for  $Ax = y$  first stretches  $A \rightarrow A^S$  and  $y \rightarrow y^S$ , then solves  $A^S z = y^S$ , and finally squeezes  $z \rightarrow z_S = x$ .

$$\begin{array}{ccc} A & & y \\ \downarrow & & \downarrow \\ A^S & z = & y^S \\ \downarrow & & \\ z_S & & \end{array}$$

The manipulative steps introduce errors beyond those of solving  $A^S z = y^S$ . Nevertheless, if the embedded solution process is stable in the customary sense, and if  $A$  is well conditioned, then the overall process accurately solves  $Ax = y$ .

The present analysis takes the standard approach toward understanding finite precision computation. The errors are interpreted as being governed by both the equations and the algorithm. *Error analyses* follow individual arithmetic errors through an algorithm and are technically demanding, but in stretching's case most errors arise in the solution of  $A^S z = y^S$  and can be assumed accounted for by other analyses. These accumulated errors are viewed as perturbing the equations rather than the solution, and the solution's accuracy is assessed by two inequalities.

The *vector perturbation inequality* emphasizes the role of the equations. Any approximate solution  $\bar{x}$  for  $Ax = y$  exactly solves  $A\bar{x} = y - r$  in which  $r$  is the *residual*  $y - A\bar{x}$ .

$$\begin{array}{l} \text{if } Ax = y \text{ and} \\ \bar{x} \text{ is any vector} \end{array} \implies \frac{\|x - \bar{x}\|}{\|x\|} \leq \kappa(A) \frac{\|r\|}{\|y\|}$$

An algorithm might produce an  $\bar{x}$  with a small relative residual (bars traditionally denote computed quantities), but no matter how small, the *condition number*

$$\kappa(A) = \|A\| \|A^{-1}\| \geq 1$$

scales the bound and perhaps the error. The bound may not be sharp because the error varies with  $A$  and  $y$  (not merely linearly with the condition number as the bound suggests). Yet the bound is valuable because the residual is directly observable and because the condition number is intrinsic to the matrix (and to the measurement of errors by norms—this and the next inequality are valid for any consistent matrix-vector norm).

The *matrix perturbation inequality* relies on details of the solution method, with the approximate solution expected to be the exact solution of an approximate problem derived from error analysis.

$$\begin{array}{l} Ax = y \\ (A + E)\bar{x} = y \end{array} \implies \frac{\|x - \bar{x}\|}{\|x\|} \leq \frac{\|A^{-1}E\|}{1 - \|A^{-1}E\|}$$

The solution algorithm determines  $\bar{x}$  from  $A$  and  $y$ , but the perturbation  $E$  may be any for which  $E\bar{x} = r$ . The bound additionally requires  $\|A^{-1}E\| < 1$ , implying  $\|E\| < \|A\|$ , and a *stable* algorithm has some  $E$  provably small relative to  $A$ , so the bound is usefully weakened as follows.

$$\frac{\|x - \bar{x}\|}{\|x\|} \leq \frac{\|A^{-1}E\|}{1 - \|A^{-1}E\|} \leq \kappa(A) \frac{\|E\|}{\|A\| - \|E\|}$$

Bounds upon some  $\|E\|$  that depend on  $A$  but not on  $y$  represent the error as being independent of  $y$  and have been derived for several algorithms. They are primarily the work of J. H. Wilkinson and are beyond the scope of this discussion. They can be found in many texts including [5] [7] and references therein.

The inequalities above are too flexible to be of predictive value when the error is small, but they diagnose the cause when the error is large. They prove stable algorithms applied to well-conditioned matrices yield accurate solutions. Section 6 illustrates the risk of calculating without a performance guaranty. There, a plausible but imperfect method is found to produce unexpectedly large errors.

#### 4a. Condition Numbers

The condition numbers of stretched matrices vary with the newly introduced nonzeros which in some sense bind the stretched matrices together. The glue lies in the submatrices  $G$  and  $D_1, D_2, \dots, D_{m-1}$  of Definitions 1 and 2. This section finds glue that favorably bounds the condition of matrices stretched by Definition 2.

The bounds are stated not for a single stretching,  $A \rightarrow A^S$ , but rather for a sequence of stretchings.

$$A \rightarrow A^S \rightarrow A^{SS} \rightarrow \dots \rightarrow A^{SS\dots S}$$

This generality anticipates sparse factorization software that might stretch many times. For example, a row

$$r = [a_1 \quad a_2 \quad a_3 \quad a_4 \quad a_5 \quad a_6]$$

could stretch once

$$r^S = \begin{bmatrix} a_1 & \cdot & \cdot & \cdot & \cdot & \cdot & - & & & \\ \cdot & a_2 & a_3 & a_4 & \cdot & \cdot & + & - & & \\ \cdot & \cdot & \cdot & \cdot & a_5 & \cdot & & + & - & \\ \cdot & \cdot & \cdot & \cdot & \cdot & a_6 & & & + & \end{bmatrix}$$

and then a descendent could stretch again.

$$r^{SS} = \begin{bmatrix} a_1 & & & & & & - & & & \\ \cdot & a_2 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & - \\ \cdot & \cdot & a_3 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & + & - \\ \cdot & \cdot & \cdot & a_4 & \cdot & \cdot & + & - & \cdot & & + \\ & & & & a_5 & & & + & - & & \\ & & & & & a_6 & & & + & & \end{bmatrix}$$

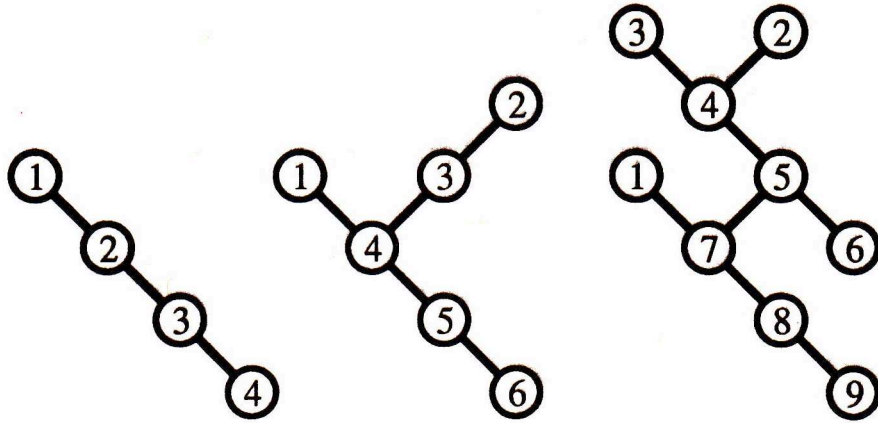
The  $\pm$ 's are the glue. Rows that contain mostly glue could stretch to rows that contain only glue, with little apparent order.

$$r^{SSS} = \begin{bmatrix} a_1 & & & & & & - & & & & \\ & a_2 & & & & & & & - & & \\ \cdot & \cdot & a_3 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & - \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & + & \cdot & + & - \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & - & & + & - \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & & & + \\ & & & a_4 & & & + & - & & & + & & & \\ & & & & a_5 & & & + & - & & & & & \\ & & & & & a_6 & & & + & & & & & \end{bmatrix}$$

Yet glue links the stretched rows or columns in a tree structure described by the following Definition.

**Definition 3, (Weighted) Row and Column Graphs.** *The row graph of a matrix takes rows for vertices and connects two by an edge of weight  $w$  if they have nonzeros in the same  $w$  columns. The row graph of matrix  $G$  is  $\text{row}(G)$ . Row graphs of submatrices are subgraphs of row graphs, and so on. The column graph is similar.<sup>3</sup>*

A simple row stretching links its stretched rows by linear trees within the row graph of its glue columns; compounded stretchings build more elaborate trees.<sup>4</sup> Figure 11 displays the successive trees for the compound stretching of a single row  $r \rightarrow r^S \rightarrow r^{SS} \rightarrow r^{SSS}$  in the text above. When many rows in a matrix stretch, then the descendents of each become separate maximally connected subgraphs in the glue's row graph, and those subgraphs are trees. The rows in each tree contain all the scattered pieces of some original row and can be summed to recover that row. For column stretchings, exchange *row* and *column* in this discussion.



**Figure 11.** Row graphs of the matrices stretched from a single row in the text of Section 4a. Loop edges have been omitted.

The trees in the row and column graphs of the glue enable the following bound on condition numbers. The bound is particularly pleasing because it depends on the maximum descendents of any one row or column—the size of the largest tree—but it does not depend on the total descendents of all rows and columns. For example, if many rows stretch in the block fashion of Definition 2, then the bound varies with the number of descendents for any one row, as though only one row stretched. In this way the bound is independent of the total growth in the size of the matrix.

**Theorem 5.** *If*

$$A \rightarrow A^S \rightarrow A^{SS} \dots \rightarrow A^{SS\dots S}$$

*is a sequence of simple row or column stretchings but not both, and if each row or column of  $A$  stretches to at most  $m$  rows or columns of  $A^{SS\dots S}$ , and if Definition 2's matrices  $D_i$  have the form  $\sigma I$  for the same  $\sigma$ , then*

<sup>3</sup> This concept isn't in [5] [6] and may be new. Other weights can be used, for example, the inner product.

<sup>4</sup> A *tree* is a connected graph that breaks in two with the loss of any non-loop edge. Equivalently, a tree has exactly one non-repeating path between every two vertices.

the following choices for  $\sigma$

$\sigma$	$p = 1$	$p = \infty$
row	$\ A\ _p/2$	$\ A\ _p$
column	$\ A\ _p$	$\ A\ _p/2$

yield a final stretched matrix  $A^{SS\cdots S}$  with bounded condition number

$$\kappa_p(A^{SS\cdots S}) \leq c \kappa_p(A)$$

in which the multiplier  $c$  is given below.

$c$	$p = 1$	$p = \infty$
row	$2m - 1$	$m^2$
column	$m^2$	$2m - 1$

When the sequence of stretchings is disjoint in the sense that later stretchings do not alter the rows or columns of earlier stretchings, then  $3m$  can replace  $m^2$  in this table. All these bounds are sharp for some matrices (proof appears in Appendix 1).

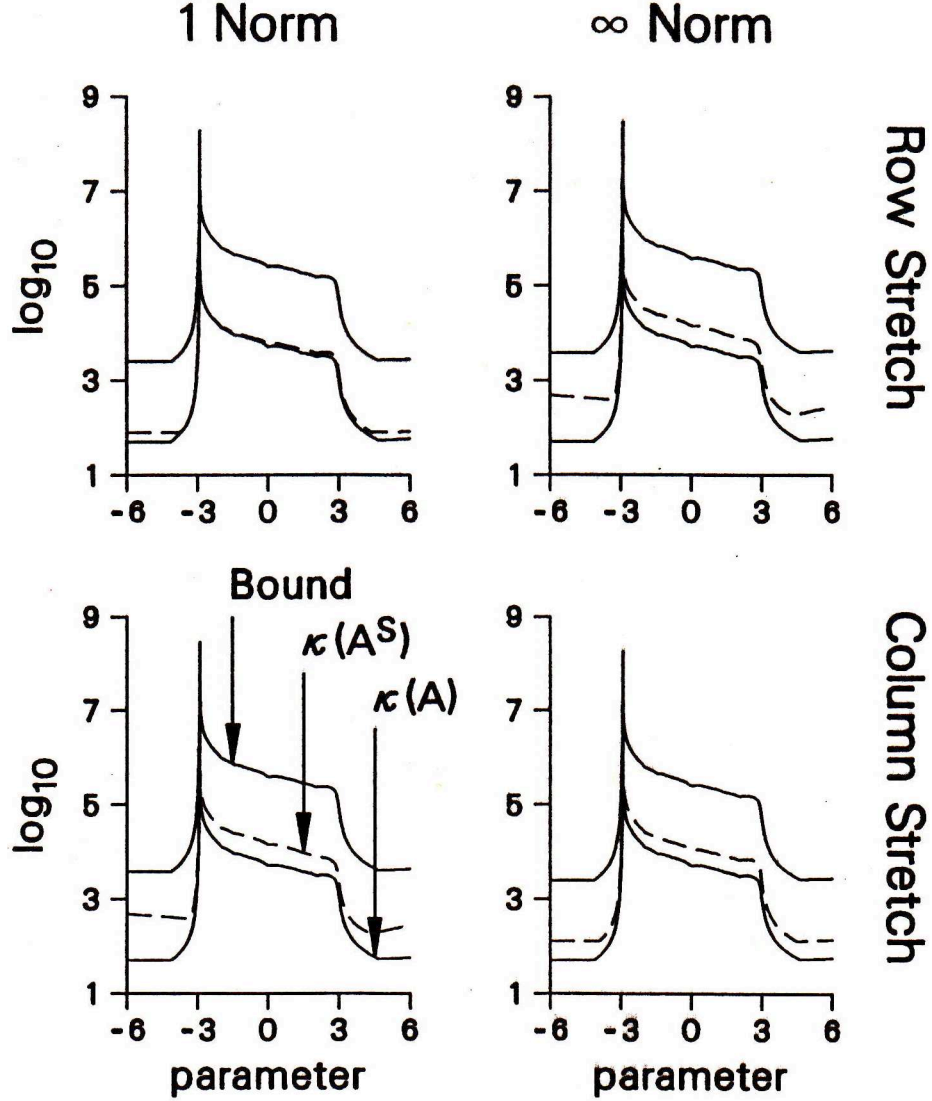
Figure 12 illustrates Theorem 5. The matrices of Figure 6 are stretched in each of the four ways indicated by the Theorem's tables. Either row or column stretching is performed, and the glue is chosen to bound either the 1-norm or the  $\infty$ -norm condition numbers. Figure 12 plots the condition numbers before and after stretching. In all cases the condition numbers increase less than the moderate bounds allow.

#### 4b. A Priori Accuracy

J. H. Wilkinson called error bounds *a priori* when they guaranty accuracy without measuring residuals and the like. He obtained a priori bounds for solving  $Ax = y$  under the two conditions discussed in Section 4. When simple stretchings are used, Theorem 5 assures  $A^S$  is well conditioned if  $A$  is, and the algorithm that solves  $A^S z = y^S$  must be stable. These are Wilkinson's conditions. Thus, the requirements for a priori accuracy are no more stringent with stretching than without.

Theorem 6 presents a priori bounds for solving  $Ax = y$  by iterated simple row or column stretching with glue chosen by Theorem 5. As a practical matter, this glue sometimes may be unnecessary. Replacing Theorem 5's  $\sigma$  with 1 rescales the columns in simple row stretching and has little effect on triangular factorization with row reordering, a popular and often stable algorithm. Nevertheless, the a priori bounds require Theorem 5's glue. The computed stretched matrix thus differs from the ideal matrix because  $\|A\|$  must be computed with imprecise machine arithmetic. Theorem 6 accounts for this discrepancy.

Elaborate vector manipulations  $y \rightarrow y^S$  and  $z \rightarrow z_S$  also generate errors, but the simplest conveniently eliminate the need for additional error analysis. As explained in Section 3, a simple row stretching admits several vector stretchings, but  $z \rightarrow z_S$  must copy entries out of  $z$ , that is, must *gather*. Conversely, simple column stretching admits several vector squeezings, but  $y \rightarrow y^S$  must copy entries into  $y^S$ , that is, must *scatter*. Both simple stretchings can accept both simple vector operations, and when they do, then forming  $A^S$  and solving  $A^S z = y^S$  are the only sources of machine arithmetic error. In this case, relative errors in  $z_S$  bound relative errors in  $z$ , and Appendix 1 combines these bounds with Theorem 5 to obtain the following result.



**Figure 12.** Condition numbers of the matrices in Figure 6 (lower solid lines) and condition numbers after stretching (dashed lines) to remove either bordering rows or columns. Theorem 5 specifies glue that bounds (upper solid lines) either the 1- or  $\infty$ -norm condition numbers. Appendix 2 and Section 4a explain the calculations.

**Theorem 6.** If  $A \rightarrow A^S$  is stretching of a nonsingular matrix obtained from a sequence of simple row or column stretchings but not both, and if glue is chosen by Theorem 5, and if the stretched matrix  $\overline{A^S}$  is computed in finite precision arithmetic with unit roundoff  $\epsilon$ , and if the vector operations used to solve linear equations are error-free scatter  $y \rightarrow y^S$  and gather  $z \rightarrow z_S$  operations, and if the approximate solution  $\bar{z}$  to the computed stretched equations  $\overline{A^S}z = y^S$  exactly satisfies some perturbed equations  $(\overline{A^S} + E)\bar{z} = y^S$ , then

$$\delta_1 := c_1 [(1 + \epsilon)^n - 1] < 1 \quad \text{and} \quad \delta_2 := \frac{\|E\|_p}{\|\overline{A^S}\|_p} < \frac{1 - \delta_1}{1 + \delta_1}$$

imply

$$\frac{\|x - \bar{z}_S\|_p}{\|x\|_p} < c_2 \kappa(A) \frac{(\delta_1 + \delta_2 + \delta_1 \delta_2)}{1 - (\delta_1 + \delta_2 + \delta_1 \delta_2)}$$

$$\approx c_2 \kappa(A) \left( c_1 n \epsilon + \|E\|_p / \|\bar{A}^S\|_p \right)$$

in which  $c_1$  and  $c_2$  are given by the tables

$c_1$	$p = 1$	$p = \infty$	$c_2$	$p = 1$	$p = \infty$
row	2	$m$	row	$(2m - 1)^2$	$m^2$
column	$m$	2	column	$m^3$	$2m - 1$

where  $n$  is the order of  $A$  and each row of  $A$  stretches to at most  $m$  rows of  $A^S$ . When the sequence of stretchings is disjoint in that later stretchings do not alter the rows or columns of earlier stretchings, then the tables can be replaced by the ones below.

$c_1$	$p = 1$	$p = \infty$	$c_2$	$p = 1$	$p = \infty$
row	2	2	row	$(2m - 1)^2$	$3m$
column	2	2	column	$3m^2$	$2m - 1$

Thus, if  $A$  is well-conditioned, if  $\epsilon$  and  $\|E\|_p / \|\bar{A}^S\|_p$  are very small, and if  $m$  and  $n$  are not excessively large, then  $\bar{z}_S$  is a good approximate solution to  $Ax = y$  (proof appears in Appendix 1).

## 5. Arrow Matrices

The important class of bordered, banded matrices demonstrates stretching's utility. These matrices have the shape of the matrix in Figure 2, namely

$$\begin{bmatrix} B & C \\ R & E \end{bmatrix}$$

in which  $B$  is banded and the bordering rows and columns  $R$  and  $C$  are dense. Bordered matrices with general, sparse  $B$  occur frequently. The banded kind of interest here sometimes are called *arrow matrices*. For them, stretching significantly improves the solving equations by triangular factorization. In the next section, stretching compares favorably even with algorithms designed specifically for bordered systems.

Stretching has applications beyond arrow matrices, but more general sparse matrices pose questions that cannot be settled by mathematical proof. Investigation of these, like other issues involving randomly sparse matrices, requires extensive comparison of examples that is beyond the scope of this paper. Arrow equations are considered because they allow precise quantification of stretching's economies. In general, only experience proves the effectiveness of sparse matrix methods.

Bordered matrices pose a significant dilemma in the use of triangular factorization methods. The row or column order usually must change to insure numerical accuracy, but when a row or column moves out of the dense border, then the factors can become completely dense. The computational complexity of the dense case is an upper bound for all matrices but a severe overestimate for many sparse ones

[5] [6]. Special reordering strategies that avoid creating new nonzeros and special data structures that manipulate only the nonzeros yield significant economies that can be precisely quantified for banded matrices and some others. Theorem 7 shows banded matrices reduce factorization complexity from  $2n^3/3$  operations to  $2\ell(\ell + u + 1)n$  in which  $\ell$  and  $u$  are the strict lower and upper bandwidths. When the matrix is bordered, however, then the pessimistic dense case cannot be ruled out and in some cases is even likely.

**Theorem 7.** *An  $n \times n$ , dense system of linear equations can be solved by triangular factorization with row reordering for stability using*

$$\begin{array}{ll} 2n^3/3 - 2n/3 & \text{arithmetic operations for the factorization and} \\ 2n^2 - n & \text{operations for the solution phase.} \end{array}$$

*However, if the matrix is banded with strict lower and upper bandwidths  $\ell$  and  $u$ , and if  $\ell + u < n$ , then the operations reduce to*

$$\begin{array}{ll} 2\ell(\ell + u + 1)n - \ell(4\ell^2 + 6\ell u + 3u^2 + 6\ell + 3u + 2)/3 & \text{for the factorization and} \\ (4\ell + 2u + 1)n - (2\ell^2 + 2\ell u + u^2 + 2\ell + u) & \text{for the solution phase} \end{array}$$

*(proof appears in Appendix 1).*

Stretching eliminates the possibility of catastrophe for bordered, banded matrices by eliminating the border. With the customary row reordering it is sufficient to remove only the dense rows. This can be done by the simple row stretching of Definition 2. A row and column reordering then gives the stretched matrix a banded structure for which factorization with row reordering is clearly efficient. Both the stretching and the reordering depend on the following blocking of the rows and columns.

**Theorem 8.** *This row and column partitioning makes a banded matrix into a block-bidiagonal one. For a matrix of order  $n$  with strict lower and upper bandwidths  $\ell$  and  $u$ , and with  $0 < \ell + u < n$ , the columns and rows partition into blocks of the following size.*

$$\begin{array}{ll} \text{columns} & a + u, \quad \ell + u, \quad \dots, \quad \ell + u, \quad \ell + c \\ \text{rows} & a, \quad u + \ell, \quad u + \ell, \quad \dots, \quad u + \ell, \quad c \\ & 0 \leq a \leq \ell \quad 0 \leq c \leq u \quad 0 < a + c \end{array}$$

*The block-column dimension is  $m = \lceil n/(\ell + u) \rceil$ , and the block-row dimension is  $m + 1$  or  $m$  (since one of  $a$  or  $c$  may be zero). Moreover, the upper diagonal blocks are lower triangular and the lower diagonal blocks are upper triangular (proof appears in Appendix 1).*

The partitioning of Theorem 8 applied to the banded portion of an arrow matrix results in a blocked matrix.

$$\begin{bmatrix} B & C \\ R & E \end{bmatrix} = \begin{bmatrix} L_1 & & & & C_1 \\ U_1 & L_2 & & & C_2 \\ & U_2 & L_3 & & C_3 \\ & & U_3 & \ddots & \vdots \\ & & & \ddots & L_m & C_m \\ & & & & U_m & C_{m+1} \\ R_1 & R_2 & R_3 & \cdots & R_m & E \end{bmatrix}$$

The row blocks containing  $L_1$  and  $U_m$  have row orders  $a$  and  $c$  which the Theorem allows to be zero, but no harm results from including null blocks in the display. Both the picture and the Theorem assume  $B$  is square, in other words, the bordering rows number the same as the bordering columns, and the banded portion ends as shown where the bordering rows and columns intersect. Simple row stretching replaces this matrix by

$$\begin{bmatrix} L_1 & & & & & C_1 \\ U_1 & L_2 & & & & C_2 \\ & U_2 & L_3 & & & C_3 \\ & & U_3 & \ddots & & \vdots \\ & & & \ddots & L_m & C_m \\ & & & & U_m & C_{m+1} \\ R_1 & & & & & -D_1 \\ & R_2 & & & & +D_1 & -D_2 \\ & & R_3 & & & & +D_2 & \ddots \\ & & & \ddots & & & & \ddots & -D_{m-1} \\ & & & & R_m & E & & & +D_{m-1} \end{bmatrix}$$

and then applies a perfect shuffle to the blocks of rows and columns. The  $j^{th}$  block of new rows goes after the  $j^{th}$  block of old rows, and similarly for columns. The result is arresting, even beautiful.

$$\begin{bmatrix} L_1 & & & & & C_1 \\ R_1 & -D_1 & & & & 0 \\ U_1 & & L_2 & & & C_2 \\ & +D_1 & R_2 & -D_2 & & 0 \\ & & U_2 & & L_3 & C_3 \\ & & & +D_2 & R_3 & \ddots \\ & & & & U_3 & \ddots \\ & & & & & \ddots & -D_{m-1} & 0 \\ & & & & & & \ddots & L_m & C_m \\ & & & & & & & +D_{m-1} & R_m & E \\ & & & & & & & & U_m & C_{m+1} \end{bmatrix}$$

Theorem 8 goes to some trouble to insure this matrix is as good as it looks. The banded portion is seamless with uniform strict lower and upper bandwidths  $d + \ell$  and  $u$ , in which  $d$  is the depth of the border and  $\ell$  and  $u$  are the bandwidths in the original arrow matrix.

When the stretched matrix is used to solve equations, then the computational complexity varies linearly with the size of the banded portion of the original matrix, as in the purely banded case. Theorems 7 and 9 supply the following operation counts for the factorization and solution phases, respectively.

banded	additional operations when bordered and stretched
<hr/>	
$2\ell(\ell + u + 1)n + 2d(2d + 3\ell + u + 1)n + 2d(d + \ell)(2d + \ell + u + 1) \left\lceil \frac{n}{\ell + u} \right\rceil$	
$(4\ell + 2u + 1)n + 4dn + d(4d + 4\ell + 2u + 1) \left\lceil \frac{n}{\ell + u} \right\rceil$	

In these formulas,  $n + d$  is the size of the unstretched arrow matrix and  $n$  is the size of its banded portion.

**Theorem 9.** An order  $n+d$ , bordered, banded system of linear equations, whose coefficient matrix has  $d$  dense rows and columns in the bordering portion and has strict lower and upper bandwidths  $\ell$  and  $u$  in the  $n \times n$  banded portion, where  $0 < \ell + u < n$ , can be solved by simple row stretching and triangular factorization with row reordering for stability in

$$(4d^2 + 6d\ell + 2du + 2\ell^2 + 2\ell u + 2d + 2\ell)N \\ - (d + \ell)(13d^2 + 14d\ell + 12du + 4\ell^2 + 6\ell u + 3u^2 + 9d + 6\ell + 3u + 2)/3$$

arithmetic operations for the factorization and

$$(4d + 4\ell + 2u + 1)N \\ - (2d^2 + 4d\ell + 2du + 2\ell^2 + 2\ell u + u^2 + 2d + 2\ell + u)$$

operations for the solution phase, in which

$$N = n + d \left\lceil \frac{n}{\ell + u} \right\rceil$$

is the size of the stretched matrix (proof appears in Appendix 1).

## 6. Deflated Block Elimination

Bordered matrices occur with sufficient frequency to receive special treatment. Although stretching is a general sparse matrix method, it compares favorably with specialized algorithms for bordered equations. Chan's deflated block elimination [2] solves bordered, banded systems with computational complexity near stretching's, but its numerical accuracy can be much worse. Stretching therefore is more reliable for arrow matrices, and also more versatile. Deflated block elimination may be suited for other contexts, however, and should not be judged solely by this comparison.

Pure block elimination employs a representation of the inverse

$$\begin{bmatrix} B & C \\ R & E \end{bmatrix}^{-1} = \begin{bmatrix} I & -B^{-1}C \\ 0 & I \end{bmatrix} \begin{bmatrix} B^{-1} & 0 \\ 0 & (E - RB^{-1}C)^{-1} \end{bmatrix} \begin{bmatrix} I & 0 \\ -RB^{-1} & I \end{bmatrix}$$

obtained by inverting the block factorization

$$\begin{bmatrix} B & C \\ R & E \end{bmatrix} = \begin{bmatrix} I & 0 \\ RB^{-1} & I \end{bmatrix} \begin{bmatrix} B & 0 \\ 0 & E - RB^{-1}C \end{bmatrix} \begin{bmatrix} I & B^{-1}C \\ 0 & I \end{bmatrix}.$$

The factorization is a recipe for applying the inverse without evaluating the factors. There are two phases. Steps in the first phase depend on the matrix alone, those in the second repeat for each right side. Table 1 lists the steps and counts the arithmetic operations to solve

$$\begin{bmatrix} B & c \\ r^t & \alpha_0 \end{bmatrix} \begin{bmatrix} v_* \\ \beta_* \end{bmatrix} = \begin{bmatrix} v_0 \\ \beta_0 \end{bmatrix}$$

in which  $r^t$  and  $c$  replace  $R$  and  $C$  to indicate a single bordering row and column. The weakness of block elimination is the need to solve equations with a coefficient

**Table 1.** *Factorization and solution phases of block elimination with operation counts.  $B$  has order  $n$  and strict lower and upper bandwidths  $\ell$  and  $u$ ,  $\ell + u < n$ . The costs of factoring  $B$  and applying  $B^{-1}$  are from Theorem 7. Terms independent of  $n$  are omitted. Section 6 provides further explanation.*

step	operations
construct a triangular factorization of $B$	$2\ell(\ell + u + 1)n$
$u_1 := B^{-1}c$	$(4\ell + 2u + 1)n$
$\alpha_1 := \alpha_0 - r^t u_1$	$2n$
$v_1 := B^{-1}v_0$	$(4\ell + 2u + 1)n$
$\beta_1 := \beta_0 - r^t v_1$	$2n$
$\beta_* := \beta_1/\alpha_1$	
$v_* := v_1 - u_1\beta_*$	$2n$

matrix,  $B$ , that can be badly conditioned or singular even when the larger matrix is neither.

Deflated block elimination [2] attempts to correct the deficiency of the original method. The rationale for the deflated algorithm appears to be the following. It has been developed for matrices with one dense row and column, that is,  $B$  must be a maximal submatrix. Maximal submatrices of nonsingular matrices have null spaces dimensioned at most 1, so if the entire matrix is well-conditioned then it is inferred either  $B$  is well-conditioned too or has a well-separated, smallest singular value. Whenever  $B^{-1}$  must be applied to a vector, a component that lies in or near the space corresponding to the smallest singular value might be separated from the product. A modification of the block factorization recipe manipulates these decomposed vectors without loss of accuracy. There results a more complicated algorithm that gives accurate results even when  $B$  has a small singular value. When not, then there is no harm in evaluating the more elaborate formulas. The weakness of the algorithm is the need to estimate the smallest singular value of  $B$  and the associated singular vectors.

Table 2 lists the steps and counts the arithmetic operations performed by the deflated algorithm to solve the same equations as Table 1. The Table makes the following choices among the algorithm’s many variations. First, several approximations might be made to the the smallest singular value and its singular vectors. Table 2 uses the “orthogonal projector” estimates in the final row of Chan’s Table 3.1 [2, p. 125]. This choice appears to be the most economical. Second, there is some apparent variation in computing the vector decompositions. Table 2 uses “Algorithm NIA” [2, p. 126] without the final step. Chan does not recognize NIA’s final step unnecessarily applies a linear transformation to a vector invariant for the transformation. He omits the step for other reasons [2, p. 130]. Third,  $B$  might be factored and  $B^{-1}$  applied by several means. Table 2 employs the Doolittle triangular decomposition with row reordering for stability, as does Chan [2, p. 130].

The numerical performance of deflated block elimination varies with the quality of approximations to small singular values and their vectors. The approximations made by Table 2 depend on “the smallest pivot having the magnitude of the smallest singular value, which is definitely not valid in general, but which is shown empirically and theoretically to be valid in practice” (paraphrasing [2, p. 124]). It is well known “there is no correlation between small pivots and ill-conditioning” [7, p. 63], but in light of Chan’s remarks it is surprising Figure 13 shows his approximations

**Table 2.** Factorization and solution phases of deflated block elimination, with cross-references to the original notation [2], and with operation counts.  $B$  has order  $n$  and strict lower and upper bandwidths  $\ell$  and  $u$ ,  $\ell + u < n$ . The costs of factoring  $B$  and applying  $B^{-1}$  are from Theorem 7. Terms independent of  $n$  are omitted. Notation  $e_k$  is column  $k$  of an identity matrix. Section 6 provides further explanation.

original	step	operations
$A$	construct a triangular factorization of $B$	$2\ell(\ell + u + 1)n$
$k$	choose $k$ , the index of the smallest pivot	$n$
$\psi$	$u_1 := B^{-t}e_k / \ B^{-t}e_k\ _2$	$(4\ell + 2u + 4)n$
	$u_2 := B^{-1}u_1$	$(4\ell + 2u + 1)n$
$\delta$	$\alpha_1 := 1/\ u_2\ _2$	$2n$
$\phi$	$u_3 := u_2\alpha_1$	$n$
$c_b$	$\alpha_2 := u_1^t c$	$2n$
$v_D$	$u_4 := B^{-1}(c - u_1\alpha_2)$	$(4\ell + 2u + 3)n$
$h_2$	$\alpha_3 := \alpha_0 - r^t u_4$	$2n$
	$\alpha_4 := r^t u_3$	$2n$
$D$	$\alpha_5 := \alpha_2\alpha_4 - \alpha_1\alpha_3$	
$c_f$	$\beta_1 := u_1^t v_0$	$2n$
$w_D$	$v_1 := B^{-1}(v_0 - u_1\beta_1)$	$(4\ell + 2u + 3)n$
$h_1$	$\beta_2 := \beta_0 - r^t v_1$	$2n$
$h_3$	$\beta_3 := \alpha_2\beta_2 - \alpha_3\beta_1$	
$h_4$	$\beta_4 := \alpha_4\beta_1 - \alpha_1\beta_2$	
$x$	$v_* := v_1 + (u_3\beta_3 - u_4\beta_4)/\alpha_5$	$4n$
$y$	$\beta_* := \beta_4/\alpha_5$	

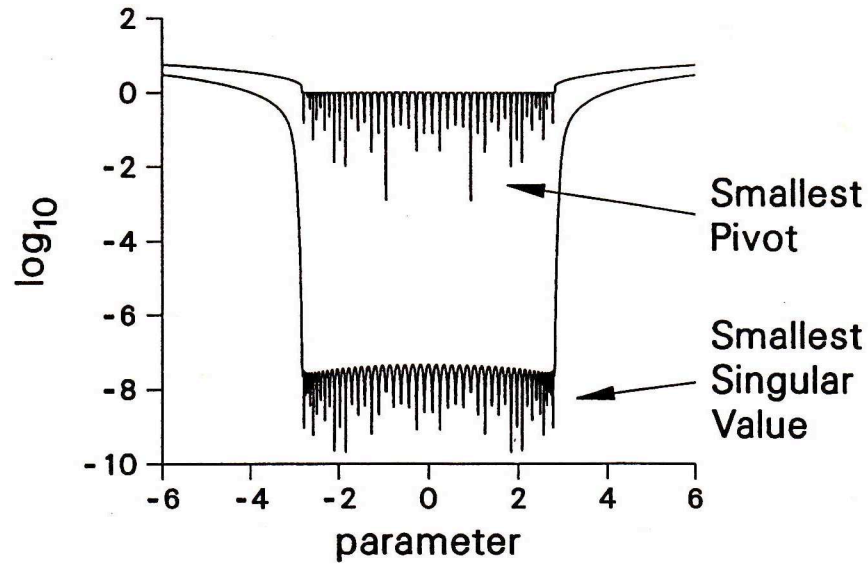
are invalid for the commonplace matrices of Figure 6. In this case deflated block elimination degenerates to pure block elimination, and Figure 14 shows the two methods have nearly identical, spectacularly large errors. Better approximations to the singular values and vectors could remedy this, but their costs would favor stretching even more.

The arithmetic costs for Table 2’s version of deflated block elimination and for simple row stretching are nearly the same, but stretching’s are generally smaller. Operations in the “factorization” phase are

$$\begin{aligned}
\text{Block Elimination} & \quad n [2\ell^2 + 2\ell u + 6\ell + 2u + 3] \\
\text{Simple Row Stretching} & \quad n [ \dots + 10\ell + 2u + 8 + \frac{6\ell + 6}{\ell + u} ] \\
\text{Deflated Block Elimination} & \quad n [ \dots + 14\ell + 6u + 18 ]
\end{aligned}$$

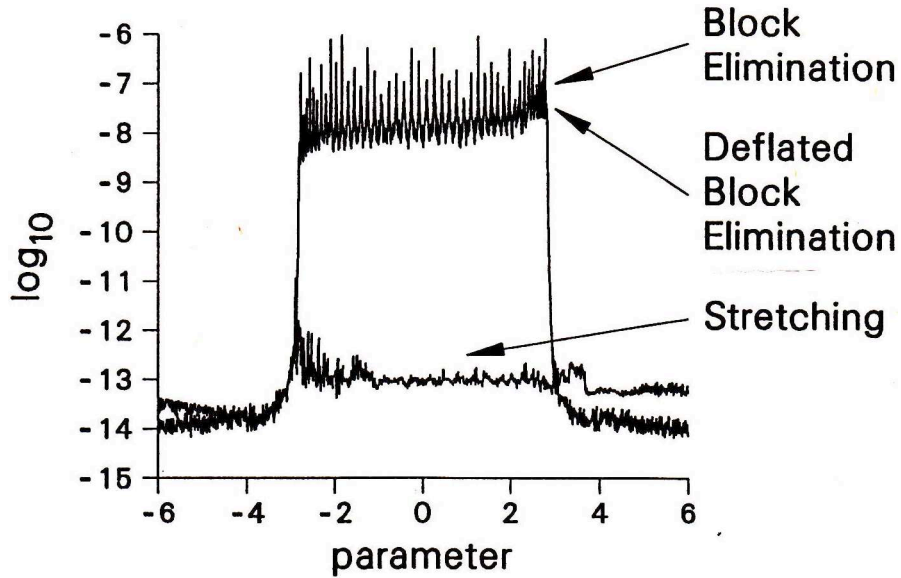
and in the “solution” phase they are

$$\begin{aligned}
\text{Block Elimination} & \quad n [4\ell + 2u + 5] \\
\text{Simple Row Stretching} & \quad n [ \dots + 9 + \frac{2\ell + 7}{\ell + u} ] \\
\text{Deflated Block Elimination} & \quad n [ \dots + 11 ]
\end{aligned}$$



**Figure 13.** *Smallest pivot and singular value for the banded portion of the matrices in Figure 6. Table 2's version of deflated block elimination assumes the pivot and singular value have the same magnitude "which is definitely not valid in general, but which is shown empirically and theoretically to be valid in practice" [2, p. 124].*

## 2-Norm Relative Error



**Figure 14.** *Maximum 2-norm relative errors for equations  $Ax = y$  with 20 different  $y$ 's solved by block elimination, deflated block elimination, and simple row stretching. The parameterized coefficient matrices  $A$  are those of Figure 6. The elimination methods cannot be distinguished at this plotting resolution. The stretching data also appears in Figure 9. Appendix 2 and Section 6 explain the calculations.*

The operation counts for the block elimination algorithms are from Tables 1 and 2. Those for simple row stretching are from Theorem 9 with the size,  $N$ , of the stretched matrix estimated as follows.

$$N = n + \left\lceil \frac{n}{\ell + u} \right\rceil \approx n \frac{\ell + u + 1}{\ell + u}$$

By this estimate stretching has the advantage in the factorization phase whenever

$$10\ell + 2u + 8 + \frac{6\ell + 6}{\ell + u} < 14\ell + 6u + 18 \iff 0 < \ell + u$$

and in the solution phase whenever

$$9 + \frac{2\ell + 7}{\ell + u} < 11 \iff 3 < u.$$

Stretching is therefore more economical than deflated block elimination for all but the smallest bandwidths.

A more significant advantage of stretching is the ability to treat arbitrarily many bordering rows and columns. Theorem 9 already reports stretching's arithmetic costs for bordered, banded matrices with many borders. In contrast, deflated block elimination has been developed for only one bordering row and column. It cannot easily be applied recursively to a succession of single borders, or be extended to multiple borders in some more direct way, because in both cases it encounters the difficult problem of submatrices with multiple small singular values.

## 7. Antecedents

Some analytic methods make some ordinary differential equations more amenable to numerical solution, and inspire matrix stretching. The following description illustrates the intellectual leap to the algebraic process and may suggest additional stretchings. This section views differential equations from the standpoint of software engineering.

For ease of notation the equations are assumed to include only 1st-order differentials and only nondifferential boundary conditions. Thus, the equations are

$$\begin{aligned} F(u, u') &= 0 \\ G(u(a)) &= 0 \quad H(u(b)) = 0 \\ F: \mathbb{R}^m \times \mathbb{R}^m &\rightarrow \mathbb{R}^m \quad (G \times H): \mathbb{R}^m \rightarrow \mathbb{R}^m \quad u: [a, b] \rightarrow \mathbb{R}^m \end{aligned}$$

in which  $F$  defines the system of  $m$  differential equations while  $G$  and  $H$  enforce the boundary conditions.  $\mathbb{R}$  is the set of real numbers.

A discrete approximate solution,  $u_k \approx u(x_k)$ , is determined at  $n + 1$  points

$$a = x_0 < x_1 < x_2 < \dots < x_n = b$$

by algebraic (or rather, nondifferential) equations.

$$\begin{aligned} F_k &:= F\left(\frac{u_k + u_{k-1}}{2}, \frac{u_k - u_{k-1}}{x_k - x_{k-1}}\right) = 0 \quad G(u_0) = 0 \quad H(u_n) = 0 \\ k &= 1, 2, \dots, n \end{aligned}$$

When these discrete equations are solved by methods like Newton's, then matrix equations must be solved to obtain the Newton corrections. The matrices are Jacobian matrices for the ensemble of functions above. Ordering the variables and equations in the natural way

$$u_0, u_1, u_2, \dots, u_n \quad G, F_1, F_2, \dots, F_n, H$$

gives the matrices a banded structure

$$J = \begin{bmatrix} A_0 & & & & & \\ B_0 & A_1 & & & & \\ & B_1 & A_2 & & & \\ & & B_2 & \ddots & & \\ & & & \ddots & A_n & \\ & & & & & B_n \end{bmatrix}$$

in which  $A_0, A_{j>0}, B_{k<n}$  and  $B_n$  are Jacobian matrices for  $G, F_j, F_{k+1}$  and  $H$  with respect to  $u_0, u_j, u_k$  and  $u_n$ , respectively. The matrix  $J$  is square because  $A_0$  and  $B_n$  may not be. For example, if all the boundary conditions are applied at the left endpoint then  $H$  and  $B_n$  are vacuous. The banded structure allows the linear equations to be solved by the efficient matrix factorization process of Theorem 7.

Arrow matrices occur when parameters and constraints accompany the differential equations

$$\begin{aligned} F(u, \lambda, u') &= 0 & 1 - \int_a^b (w^t u)^2 &= 0 \\ G(u(a)) &= 0 & H(u(b)) &= 0 \end{aligned}$$

in which  $w$  is a vector of coefficients that define the integral constraint. With the dependent parameter  $\lambda$  placed after the discrete variables, and with a discrete analogue of the constraint

$$E(u_0, u_1, u_2, \dots, u_n) := 1 - \sum_{k=1}^n (x_k - x_{k-1}) \frac{(w^t u_k)^2 + (w^t u_{k-1})^2}{2} = 0$$

placed after the other equations, the matrices acquire borders

$$J = \begin{bmatrix} A_0 & & & & & 0 \\ B_0 & A_1 & & & & c_1 \\ & B_1 & A_2 & & & c_2 \\ & & B_2 & \ddots & & \vdots \\ & & & \ddots & A_n & c_n \\ & & & & B_n & 0 \\ r_0^t & r_1^t & r_2^t & \dots & r_n^t & 0 \end{bmatrix}$$

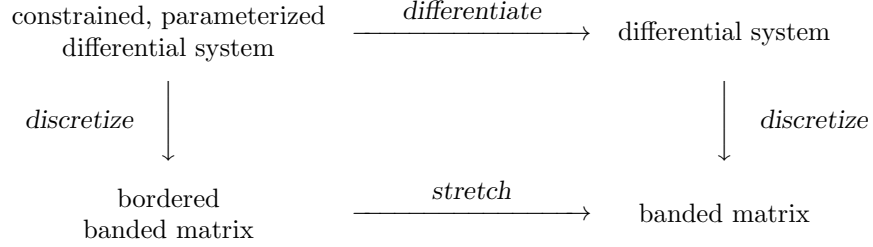
in which  $c_k$  and  $r_k^t$  are the Jacobian matrices of  $F_k$  and  $E$  with respect to  $\lambda$  and  $u_k$ , respectively. Section 5 shows arrow matrix equations can be solved efficiently after stretching to remove the border.

However, the banded structure is traditionally recovered by transforming the differential system rather the algebraic equations, as follows. Differentiating the parameter and the constraint

$$\begin{aligned} F(u, \lambda, u') &= 0 & \lambda' &= 0 & v' - \|u\|_2^2 &= 0 \\ G(u(a)) &= 0 & v(a) &= 0 & H(u(b)) &= 0 & v(b) &= 1 \end{aligned}$$

produces an un-parameterized, un-constrained differential system whose discretization again results in banded Jacobian matrices. These differential transformations are familiar simplifying devices in the theory of differential systems. They find widespread application in numerical software [8] and special mention in numerical textbooks [9, p. 47]. Simple row and column stretchings correspond to differentiating the constraint and the parameter, respectively.

The correspondence between differentiating and stretching is not precise because the discretization intervenes. The analytic transformation (differentiating) precedes the discretization, while the algebraic one (stretching) follows.



The two paths through the diagram may not lead to the same Jacobian matrix. If the discretization employs a high order approximation to  $\lambda'$ , for example, then the matrix rows for the equation  $\lambda' = 0$  unnecessarily contain more than the two non-zeroes inserted by simple column stretching. The differential transformation therefore results in a Jacobian matrix indirectly chosen and likely suboptimal. Moreover, any change to the differential system can subtly perturb the discretization. The integral constraint may yield slightly different discrete approximations in its differentiated form. In contrast, stretching guarantees efficient solution of the matrix equations—however derived—and thus allows the most appropriate discretization of the differential system.

The originality of the algebraic approach can be appreciated by considering related problems for which there is no convenient analytic interpretation, and consequently, to which nothing approximating stretching has been applied. These problems are parameterized equations whose solution is desired as a function of the parameter.

$$\begin{aligned}
 & F(u, \lambda, u') \\
 & G(u(a)) = 0 \quad H(u(b)) = 0 \\
 & F: \mathbb{R}^m \times \mathbb{R}^m \rightarrow \mathbb{R}^m \quad (G \times H): \mathbb{R}^m \rightarrow \mathbb{R}^m \quad u: [a, b] \rightarrow \mathbb{R}^m
 \end{aligned}$$

In this case the discrete equations are underdetermined

$$\begin{aligned}
 F_k &:= F\left(\frac{u_k + u_{k-1}}{2}, \lambda, \frac{u_k - u_{k-1}}{x_k - x_{k-1}}\right) = 0 \quad G(u_0) = 0 \quad H(u_n) = 0 \\
 & k = 1, 2, \dots, n
 \end{aligned}$$

and the extra variable,  $\lambda$ , gives the Jacobian matrices more columns than rows.

$$J = \begin{bmatrix} A_0 & & & & 0 \\ B_0 & A_1 & & & c_1 \\ & B_1 & A_2 & & c_2 \\ & & B_2 & \ddots & \vdots \\ & & & \ddots & A_n & c_n \\ & & & & B_n & 0 \end{bmatrix} \quad \vec{u} = \begin{bmatrix} u_0 \\ u_1 \\ u_2 \\ \vdots \\ u_n \\ \lambda \end{bmatrix}$$

Mild assumptions guarantee a differentiable curve of discrete solutions,  $\vec{u}$ , which can be traced in a variety of ways.

Keller [10] appears to be the first to consider the numerical problem of following the solution curve through  $\mathbb{R}^{n+2}$ . He locally parameterizes the curve as  $\vec{u}(\sigma)$  in which  $\sigma$  is a local approximation to arclength. The points on the curve near a known point  $\vec{u}(0)$  are specified by augmenting the discrete equations with the projection equation

$$\vec{\tau} \cdot [\vec{u}(\sigma) - \vec{u}(0)] - \sigma = 0 \quad \vec{\tau} = \begin{bmatrix} \tau_0 \\ \tau_1 \\ \tau_2 \\ \vdots \\ \tau_n \\ \mu \end{bmatrix}$$

in which  $\cdot$  is the vector dot product and the ancillary vector  $\vec{\tau}$  approximates a tangent to the curve (equivalently, a right null vector of  $J$ ) at the known solution. Given the known solution at  $\sigma = 0$ , another is found by solving the augmented equations for some  $\sigma > 0$ , then the curve is re-parameterized, and the process is repeated. Only sufficiently small  $\sigma$  can be expected to uniquely determine  $\vec{u}(\sigma)$ , and the tangent must be approximated in some way, but the details of Keller's *pseudo-arclength continuation method* are of no concern here.

If variants of Newton's method are used to solve the augmented equations, then the correction equations again feature arrow matrices.

$$J = \begin{bmatrix} A_0 & & & & 0 \\ B_0 & A_1 & & & c_1 \\ & B_1 & A_2 & & c_2 \\ & & B_2 & \ddots & \vdots \\ & & & \ddots & A_n & c_n \\ & & & & B_n & 0 \\ \tau_0^t & \tau_1^t & \tau_2^t & \dots & \tau_n^t & \mu \end{bmatrix}$$

Keller [10] suggests pure block elimination for these matrices, while Chan [2] appears to have developed deflated block elimination with this problem in mind. These methods are discussed in Section 6. A stretching process would be preferable but isn't employed—apparently because the bordered matrices are conceived in a discrete rather than an analytic context.

No doubt many devices have been used over the years to trade inconvenient matrices for more convenient, larger matrices. Until this paper, however, only the *capacitance matrix method* has received much attention. It obtains larger matrices by domain embedding of partial differential equations, and views the enlarged problems as perturbed ones susceptible to the Sherman-Morrison-Woodbury formula [7] [11]. Buzbee, Dorr, George and Golub [1] explain this approach and provide references to earlier work. The latitude in applying the Woodbury formula produces variations in numerical accuracy which continue to be of research interest [4], but the matrix enlargement process has not been generalized. The capacitance matrix method thus retains the flavor and terminology of domain embedding. Presumably, it can be recast as a matrix `STRETCHING`.

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# Appendix 1. Proofs

This appendix proves the theorems cited in the text.

**Theorem 1.** *If  $A$  and  $A^S$  are nonsingular and*

$$\begin{array}{ccc} \text{for some matrix } Y & \text{or} & \text{for some matrix } X \\ \begin{array}{l} {}^-X := A^{-1}Y A^S \\ Y^- := \text{any right inverse of } Y \end{array} & \left| \right. & \begin{array}{l} {}^-X := \text{any left inverse of } X \\ Y^- := A^S X A^{-1} \end{array} \end{array}$$

*then  $A^{-1} = {}^-X(A^S)^{-1}Y^-$ .*

**Corollary to Theorem 1.** *If in addition*

$$X := (A^S)^{-1}Y^- A \quad \left| \quad Y := A {}^-X(A^S)^{-1} \right.$$

*then  ${}^-X X = I$ ,  $Y Y^- = I$  and  $A = Y A^S X$ .*

*Proof by substitution and multiplication.* In the case parameterized by  $Y$

$$\begin{aligned} A [{}^-X(A^S)^{-1}Y^-] &= A [(A^{-1}Y A^S)(A^S)^{-1}Y^-] = Y Y^- = I \\ {}^-X X &= (A^{-1}Y A^S)[(A^S)^{-1}Y^- A] = I \\ Y A^S X &= Y A^S [(A^S)^{-1}Y^- A] = A \end{aligned}$$

and similarly in the case parameterized by  $X$ . *End of proof.*

**Theorem 2.** *If  $A \rightarrow A^S$  is a row or column stretching and if  $A$  is nonsingular, then  $A^S$  is nonsingular and  $A^{-1} = {}^-X(A^S)^{-1}Y^-$ .*

*Proof.* In the row stretching case there are matrices  $B$ ,  $G$  of full rank,  $P$  a permutation matrix,  $Y$  and  $Y^-$  with the following relationships.

$$\begin{array}{lcl} A^S = [B & G] P^t & \begin{array}{l} Y B = A \\ Y G = 0 \end{array} & \begin{array}{l} {}^-X = [I \quad 0] P^t \\ Y^- = \text{any right inverse of } Y \end{array} \end{array}$$

Suppose  $A^S u = 0$ , and let

$$P \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = u$$

in which the order of  $v_1$  is the column order of  $B$ , and the order of  $v_2$  is the column order of  $G$ . With this notation,  $B v_1 + G v_2 = A^S u = 0$  so

$$A v_1 = (Y B) v_1 = Y B v_1 - Y (B v_1 + G v_2) = -Y G v_2 = 0$$

and thus  $v_1 = 0$  because  $A$  is nonsingular. This and  $B v_1 + G v_2 = 0$  imply  $v_2 = 0$  because  $G$  has full rank. Altogether  $u = 0$  hence  $A^S$  is nonsingular. Moreover,

$$A^{-1}Y A^S = A^{-1}(Y A^S P) P^t = A^{-1}[A \quad 0] P^t = [I \quad 0] P^t = {}^-X$$

from which Theorem 1 asserts  $A^{-1} = {}^-X(A^S)^{-1}Y^-$ . The column stretching case is similar. *End of Proof.*

**Corollary to Theorem 2.** *If  $A \rightarrow A^S$  is a row or column stretching of a nonsingular matrix  $A$ , if  $\neg X$  and  $X$  are the auxiliary matrices in Definition 1, and if  $A^S z = y^S$  are the stretched equations corresponding to  $Ax = y$ , then not only  $\neg X z = x$  but also  $z = Xx$ .*

*Proof.* In the row stretched case,  $Xx = [(A^S)^{-1}Y^{\neg}A]x = (A^S)^{-1}y^S = z$ . The column case is more interesting. The Theorem says the stretched equations can be used to solve the unstretched equations by way of the identity  $\neg X z = x$  in which  $\neg X$  can be any left inverse for the  $X$  which parameterizes the stretching. Since  $\neg X(Xx) = x$ , so  $z - Xx$  lies in the right null space of every left inverse for  $X$ . Those left inverses are precisely  $(X^t X)^{-1}X^t + N$  in which  $N$  is any left annihilator of  $X$  with the proper row dimension. Let  $z - Xx = Xv_1 + v_2$  decompose  $z - Xx$  over the column space of  $X$  and its orthogonal complement.

$$0 = [(X^t X)^{-1}X^t + N](Xv_1 + v_2) = v_1 + Nv_2$$

The choice  $N = 0$  shows  $v_1 = 0$ , thus  $0 = Nv_2$  for every  $N$ . The rows of  $N$  can be any vectors in the left null space of  $X$ , which contains  $v_2^t$ . Thus  $v_2 = 0$  and altogether  $z - Xx = 0$ . *End of Proof.*

**Theorem 3.** *A simple row stretching in the sense of Definition 2 is a row stretching in the sense of Definition 1, and similarly for column stretchings.*

*Proof for row stretching.* A simple row stretching has

$$Q_1 A Q_2 = \begin{bmatrix} A_{1,1} & A_{1,2} & A_{1,3} & \dots & A_{1,m} \\ A_{2,1} & A_{2,2} & A_{2,3} & \dots & A_{2,m} \end{bmatrix}$$

$$\downarrow$$

$$Q_3 A^S Q_4 = \underbrace{\begin{bmatrix} A_{1,1} & A_{1,2} & A_{1,3} & \dots & A_{1,m} \\ A_{2,1} & & A_{2,2} & & \\ & & A_{2,3} & & \\ & & & \ddots & \\ & & & & A_{2,m} \end{bmatrix}}_{\tilde{B}} \underbrace{\begin{bmatrix} 0 & 0 & \dots & 0 \\ -D_1 & -D_2 & \dots & -D_{m-1} \\ +D_1 & +D_2 & \dots & +D_{m-1} \end{bmatrix}}_{\tilde{G}}$$

where the  $Q_i$  are permutation matrices and  $\tilde{Y} Q_3 A^S Q_4 = [Q_1 A Q_2 \quad 0]$  with

$$\tilde{Y} = \begin{bmatrix} I_1 & & & \\ & I_2 & I_2 & \dots & I_2 \end{bmatrix}$$

in which  $I_1$  and  $I_2$  are identity matrices. The permutation matrices perform the reorderings that Definition 2 allows before and after the stretching. The matrices

$$B = Q_3^t \tilde{B} Q_2^t \quad G = Q_3^t \tilde{G} \quad P = Q_4 \begin{bmatrix} Q_2^t & \\ & I \end{bmatrix} \quad Y = Q_1^t \tilde{Y} Q_3$$

are needed to give the simple row stretching the appearance of a general row stretching.  $G$  has full rank because the staircase sparsity pattern of the glue columns makes them linearly independent (the  $D_j$  are nonsingular by definition).  $P$  is a

permutation matrix.  $Y$  has full rank. Identities like those in Definition 1 follow by marshalling the permutations.

$$\begin{aligned} A^S P &= (Q_3^t [\tilde{B} \quad \tilde{G}] Q_4^t) \left( Q_4 \begin{bmatrix} Q_2^t & \\ & I \end{bmatrix} \right) = [Q_3^t \tilde{B} Q_2^t \quad Q_3^t \tilde{G}] = [B \quad G] \\ YB &= (Q_1^t \tilde{Y} Q_3) (Q_3^t \tilde{B} Q_2^t) = Q_1^t \tilde{Y} \tilde{B} Q_2^t = A \\ YG &= (Q_1^t \tilde{Y} Q_3) (Q_3^t \tilde{G}) = Q_1^t \tilde{Y} \tilde{G} = 0 \end{aligned}$$

*End of proof.*

**Theorem 4.** *If  $A \rightarrow A^S$  is a simple row or column stretching as in Definition 2, then*

$$\det A^S = \det A \prod_{j=1}^{m-1} \det D_j$$

*with perhaps a sign change when the rows and columns are reordered as the definition allows.*

*Proof.* With no reordering, simple row stretching has

$$A = \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} = \begin{bmatrix} A_{1,1} & A_{1,2} & A_{1,3} & \dots & A_{1,m} \\ A_{2,1} & A_{2,2} & A_{2,3} & \dots & A_{2,m} \end{bmatrix}$$

and

$$A^S = \begin{bmatrix} A_{1,1} & A_{1,2} & A_{1,3} & \dots & A_{1,m} & 0 & 0 & \dots & 0 \\ A_{2,1} & & & & & -D_1 & & & \\ & A_{2,2} & & & & +D_1 & -D_2 & & \\ & & A_{2,3} & & & & +D_2 & \ddots & \\ & & & \ddots & & & & \ddots & -D_{m-1} \\ & & & & A_{2,m} & & & & +D_{m-1} \end{bmatrix}.$$

Let

$$S = \begin{bmatrix} I_1 & & & & & & & & \\ & I_2 & I_2 & I_2 & \dots & I_2 & & & \\ & & I_2 & & & & & & \\ & & & I_2 & & & & & \\ & & & & \ddots & & & & \\ & & & & & I_2 & & & \end{bmatrix}$$

in which  $I_1$  and  $I_2$  are identity matrices with orders equal to the row orders of  $A_1$  and  $A_2$ . The product

$$SA^S = \begin{bmatrix} A_{1,1} & A_{1,2} & A_{1,3} & \dots & A_{1,m} & 0 & 0 & \dots & 0 \\ A_{2,1} & A_{2,2} & A_{2,3} & \dots & A_{2,m} & 0 & 0 & \dots & 0 \\ & A_{2,2} & & & & +D_1 & -D_2 & & \\ & & A_{2,3} & & & & +D_2 & \ddots & \\ & & & \ddots & & & & \ddots & -D_{m-1} \\ & & & & A_{2,m} & & & & +D_{m-1} \end{bmatrix}$$

is  $2 \times 2$  block lower triangular. The first diagonal block is  $A$  and the second is  $(m-1) \times (m-1)$  block upper triangular with diagonal blocks  $D_1, D_2, \dots, D_{m-1}$ . Thus

$$\det A^S = (\det S)(\det A^S) = \det(SA^S) = \det A \prod_{j=1}^{m-1} \det D_j.$$

*End of proof.*

**Lemma 1 to Theorem 5.** *A sequence of row stretchings is a row stretching, and similarly for column stretchings.*

*Proof for row stretching by induction on the sequence length.* Each of two row stretchings  $A \rightarrow A^S \rightarrow A^{SS}$  has matrices  $B_i$ ,  $G_i$  of full column rank,  $P_i$  a permutation matrix, and  $Y_i$  of full row rank with the following relationships.

$$\begin{aligned} A^S &= [B_1 \ G_1] P_1^t & Y_1 B_1 &= A & Y_1 G_1 &= 0 \\ A^{SS} &= [B_2 \ G_2] P_2^t & Y_2 B_2 &= A^S & Y_2 G_2 &= 0 \end{aligned}$$

Thus  $B_2 = [B_1^S \ G_1^S] P_1^t$  in which  $Y_2 B_1^S = B_1$  and  $Y_2 G_1^S = G_1$ .

$$A^{SS} = \left[ \overbrace{B_1^S}^B \ \overbrace{G_1^S \ G_2}^G \right] \overbrace{\left[ \begin{array}{c} P_1^t \\ I \end{array} \right] P_2}^P \ \overbrace{Y_1 \ Y_2}^Y B = A \quad YG = 0$$

$P$  is a permutation matrix and  $Y$  has full row rank. Suppose  $Gu = 0$ . With the rows of  $u$  blocked to match the columns of  $G$ , then

$$0 = Y_2 Gu = Y_2 [G_1^S \ G_2] \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = G_1 u_1$$

so  $u_1 = 0$  because  $G_1$  has full rank, leaving  $G_2 u_2 = 0$ , so  $u_2 = 0$  because  $G_2$  has full rank. Altogether  $u = 0$ , therefore  $G$  has full rank. *End of proof.*

**Lemma 2 to Theorem 5.** *A sequence of simple row stretchings produces a row stretching  $A \rightarrow A^S$*

$$A^S P = [B \ G] \quad YB = A \quad YG = 0$$

in which  $B$ ,  $G$ ,  $P$  and  $Y$  are as in Definition 1, and additionally (1) the entries of  $A$  scatter into  $B$  in a way that preserves columns and segregates entries from different rows, and (2)  $Y$  is a matrix of 0's and 1's with exactly one 1 per column. If the matrices  $D_j$  in the simple stretchings are diagonal, then (3) the columns of  $G$  have exactly two nonzeros and those nonzeros have equal magnitudes and opposite signs, (4) non-loop edges in  $\text{row}(G)$  have weight 1, (5) the maximally connected subgraphs of  $\text{row}(G)$  are trees, and (6) the nonzeros in each row of  $Y$  pick out one maximally connected subgraph of  $\text{row}(G)$ . For column stretchings, replace  $Y$  by  $X$ , replace the equations by

$$PA^S = \begin{bmatrix} B \\ G \end{bmatrix} \quad BX = A \quad GX = 0$$

and exchange row and column in the text.

*Proof for row stretching.* Some parts of this omnibus Lemma mightn't need proof, but the whole is more easily argued together. Simple row stretchings are row stretchings (Theorem 3), and sequences of row stretchings are row stretchings (Lemma 1), so the matrices  $B$ ,  $G$ ,  $P$  and  $Y$  exist as required by Definition 1.

(1) A simple row stretching  $A \rightarrow A^S$  copies entries of  $A$  to  $A^S$ . Entries from different columns (or rows) go to different columns (respectively, rows), and those from the same column go to the same column. The stretching therefore preserves

columns but only segregates rows. Moreover, it places glue in separate new columns. An iterated stretching

$$A = A_0 \rightarrow A_0^S = A_1 \rightarrow A_1^S = A_2 \rightarrow A_2^S = A_3 \cdots A_q = A^S$$

merely copies the entries of  $A$  and the accumulating glue several times.

(2) For a simple row stretching,  $Y$  is a matrix of 0's and 1's with exactly one 1 per column (see the proof of Theorem 3). This distinctive sparsity pattern is inherited by the product of such matrices. For a sequence of row stretchings, the overall  $Y$  is the product of the  $Y_i$  for each stretching (see the proof of Lemma 1).

(3) When the  $D_j$  are diagonal, a simple row stretching  $A \rightarrow A^S$  places exactly two pieces of glue with identical magnitudes and opposite signs in new columns of  $A^S$ . Subsequent stretchings preserve columns, and while they may rearrange old columns of glue, they add nothing to them.

(4) Non-loop edges in  $\text{row}(G)$  have weight 1 because each glue column has exactly two nonzeros, and no two columns have the same sparsity pattern, so each column is the unique edge between two rows. If two columns were to have the same sparsity pattern, then they'd be linearly dependent, and  $G$  couldn't have full rank.

(5) Discarding reorderings, the glue columns of a simple row stretching  $A \rightarrow A^S$  are those new columns containing the  $D_j$ .

$$\begin{array}{c} \begin{bmatrix} A_{1,1} & A_{1,2} & \cdots & A_{1,m} \\ A_{2,1} & A_{2,2} & \cdots & A_{2,m} \end{bmatrix} \\ \downarrow \\ \begin{bmatrix} A_{1,1} & A_{1,2} & \cdots & A_{1,m} & 0 & \cdots & 0 \\ A_{2,1} & & & & -D_1 & & \\ & A_{2,2} & & & +D_1 & \ddots & \\ & & \ddots & & & \ddots & -D_{m-1} \\ & & & A_{2,m} & & & +D_{m-1} \end{bmatrix} \end{array}$$

When the  $D_j$  are diagonal the glue is better viewed after shuffling rows and columns to group entries from the same diagonal position

$$\begin{bmatrix} 0 & & & & \\ & T_1 & & & \\ & & T_2 & & \\ & & & \ddots & \\ & & & & T_k \end{bmatrix} \quad \text{in which } T_i = \begin{bmatrix} -d_{i,1} & & & & \\ +d_{i,1} & -d_{i,2} & & & \\ & +d_{i,2} & \ddots & & \\ & & \ddots & -d_{i,m-1} & \\ & & & +d_{i,m-1} & \end{bmatrix}$$

where  $k$  is the order of the  $D_j$  and  $d_{i,j}$  is the  $i$ -th diagonal entry of  $D_j$ . The row graph of each  $T_i$  is a tree (a linear tree with two leaves and no branches). No two  $T_i$  overlap in columns of  $G$ , so the rows containing each  $T_i$  are a maximally connected subgraph of  $\text{row}(G)$ . Any other row of  $G$  is zero, hence connected to nothing, hence a maximally connected subgraph and a trivial tree.

Moreover, each row of  $A$  that stretches has its own  $T_i$  in  $G$ , and each row of  $A$  that doesn't stretch has its own zero row in  $G$ , so altogether, the rows of  $A$  number the same as the maximally connected subgraphs in  $\text{row}(G)$ .

For a sequence of simple row stretchings, all but the last can be collapsed to one stretching, leaving  $A \rightarrow A^S \rightarrow A^{SS}$  in which, by induction hypothesis, the maximally connected subgraphs in the row graph of the glue columns of  $A^S$  are trees. With a suitable ordering, the last stretching  $A^S \rightarrow A^{SS}$  changes the glue

columns as follows.

$$\begin{bmatrix} G_{1,1} & G_{1,2} & \cdots & G_{1,m} \\ G_{2,1} & G_{2,2} & \cdots & G_{2,m} \end{bmatrix} \downarrow \begin{bmatrix} G_{1,1} & G_{1,2} & \cdots & G_{1,m} & 0 & \cdots & 0 \\ G_{2,1} & & & & -D_1 & & \\ & G_{2,2} & & & +D_1 & \ddots & \\ & & \ddots & & & \ddots & -D_{m-1} \\ & & & G_{2,m} & & & +D_{m-1} \end{bmatrix}$$

Some of the glue blocks  $G_{i,j}$  may have zero column dimension because the stretching  $A^S \rightarrow A^{SS}$  needn't copy old glue to every newly stretched row. For example, if no row of  $A^S$  containing old glue stretches, then the glue columns of  $A^{SS}$  would be

$$\begin{bmatrix} G_{1,1} & 0 & \cdots & 0 \\ & -D_1 & & \\ & +D_1 & \ddots & \\ & & \ddots & -D_{m-1} \\ & & & +D_{m-1} \end{bmatrix}$$

but there is no harm in allowing null blocks into the original display. As before, the glue is better viewed after shuffling rows and columns

$$\begin{bmatrix} B_0 & & & \\ B_1 & T_1 & & \\ B_2 & & T_2 & \\ \vdots & & & \ddots \\ B_k & & & T_k \end{bmatrix} \text{ in which } \begin{cases} B_0 = [G_{1,1} & G_{1,2} & \cdots & G_{1,m}] \\ B_i = \begin{bmatrix} g_{i,1} & & & \\ & g_{i,2} & & \\ & & \ddots & \\ & & & g_{i,m} \end{bmatrix} \end{cases}$$

where now  $g_{i,j}$  is the  $i$ -th row of  $G_{2,j}$ , and the  $T_i$  are as pictured earlier. When a glue-bearing row of  $A^S$  stretches

$$[g_{i,1} \quad g_{i,2} \quad \cdots \quad g_{i,m}] \longrightarrow B_i = \begin{bmatrix} g_{i,1} & & & \\ & g_{i,2} & & \\ & & \ddots & \\ & & & g_{i,m} \end{bmatrix}$$

the effect on the row graph is to replace one vertex by several, dividing the former's edges among the latter, thus breaking one maximally connected subgraph into smaller trees (the original maximally connected subgraphs are trees by the induction hypothesis, and branches of trees are trees), and finally grafting all the branches onto the new linear tree created by  $T_i$ . Thus, the old trees merely rearrange and grow to form new trees—of the same number. That the rows of  $A$  number the same as the maximally connected subgraphs of  $\text{row}(G)$  is needed below.

(6) In the multiplication  $YG = 0$ , only the two rows that contain a glue column's two nonzeros can be summed to annihilate that column's glue. The multiplication therefore sums entire maximally connected subgraphs in  $\text{row}(G)$ , that is, the nonzeros in each row of  $Y$  pick out whole maximally connected subgraphs in  $\text{row}(G)$ . Each row of  $Y$  must pick out at least one maximally connected subgraph because  $Y$  has no zero rows ( $Y$  has full row rank). Each maximally connected subgraph must be chosen by some row of  $Y$  because  $Y$  has no zero columns (columns of  $Y$  have one nonzero apiece). The rows of  $Y$  number the same as the rows of  $A$ , and as explained in the proof of (5) above, the rows of  $A$  number the same as the maximally connected subgraphs in  $\text{row}(G)$ . Therefore, each row of  $Y$  picks out exactly one maximally connected subgraph. *End of proof.*

**Lemma 3 to Theorem 5.** *If the row graph of a matrix is a tree whose non-loop edges have weight 1, and if each column has exactly 2 nonzeros, then the removal of any row leaves a nonsingular matrix. If the nonzeros in the matrix are  $\pm 1$ , then the nonzeros in the inverse are  $\pm 1$ . The column dimension therefore bounds the 1-norm and  $\infty$ -norm of the inverse.*

*Proof by induction on the number of columns.* If there is one column, then the column's two nonzeros must connect all rows (since the row graph is a tree), so there are just two rows. Removing any row leaves a nonsingular matrix and so on.

The inductive step uses the following observations. (1) Discarding any leaf-row and its column-edges makes a smaller matrix that satisfies the Lemma's hypotheses (the graph remains a tree because only a leaf is lost; each remaining column has exactly two nonzeros because columns that lost nonzeros with the removed row are removed too). (2) A leaf-row has exactly one nonzero (columns have two nonzeros so the nonzeros in a row must connect to other rows; edges have weight one so the nonzeros in a row connect to separate neighbors; a leaf has just one neighbor hence no more than one nonzero). These facts supports many inductive proofs, for example (3) the matrices described by the Lemma are square but for one extra row (if the matrix has one column then it has been observed to have two rows; if it has more than one column then removing a leaf row and its single column edge leaves a smaller matrix like the original).

The proof's inductive step has two cases. First, the removed row neighbors every other. In this case the columns of the surviving matrix have one nonzero apiece, and these lie in distinct rows because all non-loop edges in the original row graph have weight 1. This means the new, square matrix has the sparsity pattern of a permutation matrix. Its inverse is formed by transposing and reciprocating.

Second, the removed row doesn't neighbor every other. Some path from the row therefore ends at a non-neighboring leaf. With the row to be removed placed last and with the non-neighboring leaf-row and the leaf's single column-edge placed first, the matrix has the form

$$\begin{bmatrix} \sigma & 0 \\ c & B \\ 0 & r^t \end{bmatrix}$$

in which the number  $\sigma$  is not zero and the column vector  $c$  has one nonzero entry. The induction hypothesis applied to

$$\begin{bmatrix} B \\ r^t \end{bmatrix}$$

makes  $B$  nonsingular so

$$\begin{bmatrix} \sigma & 0 \\ c & B \end{bmatrix}$$

is nonsingular too. If the nonzeros are  $\pm 1$  then

$$\begin{bmatrix} \pm 1 & 0 \\ c & B \end{bmatrix}^{-1} = \begin{bmatrix} \pm 1 & \\ \mp B^{-1}c & B^{-1} \end{bmatrix}$$

in which  $\mp B^{-1}c$  is  $\pm$  a column of  $B^{-1}$ . Again by the induction hypothesis, the nonzeros in the inverse must be  $\pm 1$ . *End of proof.*

**Lemma 4 to Theorem 5.** *If  $A \rightarrow A^S$  is a row stretching produced by a sequence of simple row stretchings, and if*

$$A^S P = [B \quad G] \quad YB = A \quad YG = 0$$

are as in Definition 1, and if  $m$  is the size of the largest maximally connected subgraph in  $\text{row}(G)$ , and if  $Q$  is a permutation matrix that places any member of the glue's maximally connected subgraph for row  $j$  of  $A$  into row  $j$  of  $A^S$  (resulting in the following blockings)

$$QB = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \quad QG = \begin{bmatrix} G_1 \\ G_2 \end{bmatrix} \quad YQ^t = \begin{bmatrix} I & Y_2 \end{bmatrix}$$

then  $G_2$  is invertible. If  $A$  is nonsingular, then there is an explicit representation for  $(A^S)^{-1}$ .

$$P^t(A^S)^{-1}Q^t = \begin{bmatrix} A^{-1} & A^{-1}Y_2 \\ -G_2^{-1}B_2A^{-1} & G_2^{-1}(I - B_2A^{-1}Y_2) \end{bmatrix}$$

Moreover, if all the nonzero entries of  $G$  are  $\pm\sigma$  for the same  $\sigma$ , then  $\|G_2^{-1}B_2\|$  has the following bounds.

$$\|G_2^{-1}B_2\|_1 \leq (m-1)\|A\|_1/|\sigma| \quad \text{and} \quad \|G_2^{-1}B_2\|_\infty \leq \|A\|_\infty/|\sigma|$$

*Proof.*  $Y$  is a matrix of 0's and 1's with exactly one 1 per column (part 2 of Lemma 2), and the nonzeros in each row of  $Y$  pick out a maximally connected subgraph of  $\text{row}(G)$  (part 6), so the chosen row ordering places an identity matrix in the first block of  $YQ^t$  as shown.

The columns of  $G$  have exactly two nonzeros (part 3 of Lemma 2), non-loop edges in  $\text{row}(G)$  have weight 1 (part 4), and the maximally connected subgraphs of  $\text{row}(G)$  are trees (part 5). Each maximally connected component of  $\text{row}(G)$  therefore satisfies Lemma 3's hypotheses. Any reordering of  $G$  that groups the maximally connected subgraphs of  $\text{row}(G)$  and their column edges makes  $G$  block diagonal.  $G_2$  is obtained by removing one row from each block, so  $G_2$  is nonsingular (Lemma 3), and in the reordering described,  $G_2$  is block diagonal with blocks no larger than  $m-1$ .

If the nonzero entries of  $G$  are  $\pm\sigma$ , then those of  $G_2^{-1}$  are  $\pm 1/\sigma$  (Lemma 3), and in view of the blocking described above, no more than  $m-1$  nonzeros populate any row or column of  $G_2$ . The 1-norm bound follows from this and the fact each column of  $B_2$  has entries copied from only one column of  $A$  (part 1 of Lemma 4).

$$\|G_2^{-1}B_2\|_1 \leq \|G_2^{-1}\|_1 \|B_2\|_1 \leq |1/\sigma|(m-1)\|A\|_1$$

The rows of  $B$  segregate entries from different rows of  $A$  (part 1 of Lemma 4), so the product  $G_2^{-1}B_2$  only sums rows stretched from the same row of  $A$ . Thus, each row of the product partly reassembles some row of  $A$ , perhaps with different signs, and has  $\infty$ -norm bounded by  $\|A\|_\infty/|\sigma|$ .

Finally, the identities

$$\begin{aligned} A &= YB = (YQ^t)(QB) = B_1 + Y_2B_2 \\ 0 &= YG = (YQ^t)(QG) = G_1 + Y_2G_2 \end{aligned}$$

allow  $QA^SP$  to be written succinctly

$$QAP = \begin{bmatrix} A - Y_2B_2 & -Y_2G_2 \\ B_2 & G_2 \end{bmatrix}$$

so the formula for the inverse can be verified by multiplication. *End of proof.*

**Theorem 5.** *If*

$$A \rightarrow A^S \rightarrow A^{SS} \dots \rightarrow A^{SS\dots S}$$

*is a sequence of simple row or column stretchings but not both, and if each row or column of  $A$  stretches to at most  $m$  rows or columns of  $A^{SS\dots S}$ , and if Definition 2's matrices  $D_i$  have the form  $\sigma I$  for the same  $\sigma$ , then the following choices for  $\sigma$*

$\sigma$	$p = 1$	$p = \infty$
row	$\ A\ _p/2$	$\ A\ _p$
column	$\ A\ _p$	$\ A\ _p/2$

*yield a final stretched matrix  $A^{SS\dots S}$  with bounded condition number*

$$\kappa_p(A^{SS\dots S}) \leq c \kappa_p(A)$$

*in which the multiplier  $c$  is given below.*

$c$	$p = 1$	$p = \infty$
row	$2m - 1$	$m^2$
column	$m^2$	$2m - 1$

*When the sequence of stretchings is disjoint in the sense that later stretchings do not alter the rows or columns of earlier stretchings, then  $3m$  can replace  $m^2$  in this table. All these bounds are sharp for some matrices.*

*Proof.* Only row stretchings need be considered because column stretchings are the transpose. Moreover, multiple stretchings can be treated simultaneously because the Lemmas have done the dirtiest work. A straightforward proof remains. It blocks the columns of  $A^{SS\dots S}$  to bound  $\|A^{SS\dots S}\|$ , it blocks the rows of  $A^{SS\dots S}$  to bound  $\|(A^{SS\dots S})^{-1}\|$ , and then it minimizes the product of the bounds.

Order the columns of  $A^{SS\dots S}$  as in Definition 1 so the entries of  $A$  lie in their original columns and  $\pm\sigma$  lie in the others.

$$A^{SS\dots S}P = [B \quad \sigma G]$$

The nonzeros in each column of  $B$  are exactly the nonzeros in the same column of  $A$ . The rows have been stretched however, so the nonzeros in each row only lie among the nonzeros in some row of  $A$ . Thus

$$\|B\|_1 = \|A\|_1 \quad \text{and} \quad \|B\|_\infty \leq \|A\|_\infty.$$

Simple row stretchings build the glue columns,  $\sigma G$ , by placing one  $\pm\sigma$  pair per column. If the stretchings in the sequence do not alter rows stretched earlier, then each row of  $G$  acquires at most 2 nonzero entries. Otherwise, the nonzeros in a row of  $G$  could number as large as  $m - 1$ . Thus

$$\|G\|_1 = 2 \quad \text{and} \quad \|G\|_\infty \leq 2 \text{ or } m - 1$$

and the following bounds on  $\|A^{SS\dots S}\|$  have been easily obtained.

$$\|A^{SS\dots S}\|_1 \leq \max\{\|A\|_1, 2\sigma\} \quad \|A^{SS\dots S}\|_\infty \leq \|A\|_\infty + (2 \text{ or } m - 1)\sigma$$

The bounds on  $\|(A^{SS\cdots S})^{-1}\|$  are more subtle.

Retain the column blocking, above, and additionally order the rows as in Lemma 4 to place any member of the glue's maximally connected subgraph for row  $j$  of  $A$  into row  $j$  of  $A^{SS\cdots S}$ . Place the other stretched rows in a second block.

$$QA^{SS\cdots S}P = \begin{bmatrix} B_1 & \sigma G_1 \\ B_2 & \sigma G_2 \end{bmatrix}$$

$$YQ^t = [I \quad Y_2] \quad YA^{SS\cdots S}P = [A \quad 0]$$

$Y$  is the matrix of Definition 1 and Lemma 2 that reverses the stretching.  $Y_2$  contains only 0's and 1's, and row  $j$  of  $Y_2$  picks out the other members of the subgraph for row  $j$  of  $A$ , hence  $\|Y_2\|_\infty \leq m-1$ . Lemma 4 proves the following.

$$P^t(A^{SS\cdots S})^{-1}Q^t = \begin{bmatrix} A^{-1} & A^{-1}Y_2 \\ -\sigma^{-1}G_2^{-1}B_2A^{-1} & \sigma^{-1}G_2^{-1}(I - B_2A^{-1}Y_2) \end{bmatrix}$$

$$\|G_2^{-1}B_2\|_1 \leq (m-1)\|A\|_1 \quad \text{and} \quad \|G_2^{-1}B_2\|_\infty \leq \|A\|_\infty$$

The present notation differs from the Lemma's because  $\sigma$  is implicit there ( $G_2$  in Lemma 4 is  $\sigma G_2$  here).

At this point the proof divides into separate cases for each norm. The 1-norm of the first block-column of  $(A^{SS\cdots S})^{-1}$  is bounded by

$$\|A^{-1}\|_1 + \|\sigma^{-1}G_2^{-1}B_2A^{-1}\|_1$$

$$\leq \|A^{-1}\|_1 + \sigma^{-1}(m-1)\|A\|_1\|A^{-1}\|_1.$$

The more complicated second block-column doesn't need separate attention because the row ordering for  $A^{SS\cdots S}$  can place any row in the first block of  $A^{SS\cdots S}$ , and thus can place any column in the first block of  $(A^{SS\cdots S})^{-1}$ . The bound above therefore applies to all columns of  $(A^{SS\cdots S})^{-1}$  and so to  $\|(A^{SS\cdots S})^{-1}\|_1$ .

The product of the bounds on  $\|A^{SS\cdots S}\|_1$  and  $\|(A^{SS\cdots S})^{-1}\|_1$  is a maximum of two functions, one increasing with  $\sigma$  and the other decreasing.

$$\|A^{SS\cdots S}\|_1\|(A^{SS\cdots S})^{-1}\|_1$$

$$\leq \max\{\|A\|_1, 2\sigma\} \times \left[ \|A^{-1}\|_1 + \sigma^{-1}(m-1)\|A\|_1\|A^{-1}\|_1 \right]$$

$$= \max \begin{cases} \|A\|_1\|A^{-1}\|_1 + \sigma^{-1}(m-1)\|A\|_1^2\|A^{-1}\|_1 \\ 2\sigma\|A^{-1}\|_1 + 2(m-1)\|A\|_1\|A^{-1}\|_1 \end{cases}$$

The minimax with respect to  $\sigma$  occurs where the two functions match. The bound on  $\|A^{SS\cdots S}\|_1$  indicates this point is  $\sigma = \|A\|_1/2$  where the bound on  $\kappa_1(A^{SS\cdots S})$  is  $(2m-1)\kappa_1(A)$ . This bound is sharp for the following simple stretching of the  $1 \times 1$  identity matrix to an  $m \times m$  matrix.

$$[1]^S = \begin{bmatrix} 1 & -0.5 & & \\ & +0.5 & \ddots & \\ & & \ddots & -0.5 \\ & & & +0.5 \end{bmatrix} \quad \text{and} \quad ([1]^S)^{-1} = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ & 2 & \cdots & 2 \\ & & \ddots & \vdots \\ & & & 2 \end{bmatrix}.$$

The  $\infty$ -norm case requires a closer look at  $(A^{SS\cdots S})^{-1}$ . The norm of the first block-row is easily bounded by

$$\|A^{-1}\|_\infty + \|A^{-1}Y_2\|_\infty \leq m\|A^{-1}\|_\infty$$

because  $\|Y_2\|_\infty \leq m - 1$ . The norm of the lower left block is cleverly bounded by

$$\| -\sigma^{-1}G_2^{-1}B_2A^{-1}\|_\infty \leq \sigma^{-1}\|A\|_\infty\|A^{-1}\|_\infty$$

using an inequality supplied by Lemma 4. As in the 1-norm case, the columns in the left block of  $(A^{SS\dots S})^{-1}$  correspond to a specific choice of representatives for connected components in the row graph of  $G$ . Each column has at most  $m$  choices, so the  $\infty$ -norm of the entire lower block-row of  $(A^{SS\dots S})^{-1}$  can't exceed  $m\sigma^{-1}\|A\|_\infty\|A^{-1}\|_\infty$ .

$$\|(A^{SS\dots S})^{-1}\|_\infty \leq \max\{m\|A^{-1}\|_\infty, m\sigma^{-1}\|A\|_\infty\|A^{-1}\|_\infty\}$$

Repeating the earlier argument, the product of the bounds on  $\|A^{SS\dots S}\|_\infty$  and  $\|(A^{SS\dots S})^{-1}\|_\infty$  is a maximum of two functions, one increasing with  $\sigma$  and the other decreasing.

$$\begin{aligned} & \|A^{SS\dots S}\|_\infty \|(A^{SS\dots S})^{-1}\|_\infty \\ & \leq (\|A\|_\infty + \alpha\sigma) \times \max\{m\|A^{-1}\|_\infty, m\sigma^{-1}\|A\|_\infty\|A^{-1}\|_\infty\} \\ & = \max\left\{\begin{array}{l} m\|A\|_\infty\|A^{-1}\|_\infty + \alpha\sigma m\|A^{-1}\|_\infty \\ m\sigma^{-1}\|A\|_\infty^2\|A^{-1}\|_\infty + \alpha m\|A\|_\infty\|A^{-1}\|_\infty \end{array}\right\} \end{aligned}$$

If the stretchings in the sequence are disjoint, then the  $\alpha$  in this formula is 2, but if they operate on each others' rows, then  $\alpha$  is  $m - 1$ . In any case, the minimax with respect to  $\sigma$  occurs where the two functions match. The bound on  $\|(A^{SS\dots S})^{-1}\|_\infty$  indicates this is  $\sigma = \|A\|_\infty$  where the bound on  $\kappa_\infty(A^{SS\dots S})$  is  $(\alpha + 1)m\kappa_\infty(A)$ . When  $\alpha = m - 1$  this bound is sharp for the following iterated stretching of the  $1 \times 1$  negative identity matrix to an  $m \times m$  matrix.

$$[-1]^{SS\dots S} = \begin{bmatrix} -1 & -1 & -1 & \cdots & -1 \\ & +1 & & & \\ & & +1 & & \\ & & & \ddots & \\ & & & & +1 \end{bmatrix} \quad \text{and} \quad ([-1]^{SS\dots S})^{-1} = [-1]^{SS\dots S}$$

When  $\alpha = 2$  the bound is sharp for the following simple stretching of the  $1 \times 1$  negative identity matrix to an  $m \times m$  matrix.

$$[-1]^S = \begin{bmatrix} 0 & -1 & & & \\ -1 & +1 & -1 & & \\ & & +1 & \ddots & \\ & & & \ddots & -1 \\ & & & & +1 \end{bmatrix} \quad \text{and} \quad ([-1]^S)^{-1} = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 \\ & & 1 & \cdots & 1 \\ & & & \ddots & \vdots \\ & & & & 1 \end{bmatrix}$$

*End of proof.*

**Lemma 1 to Theorem 6.** *If  $A \rightarrow A^S$  is a row or column stretching of a nonsingular matrix obtained from a sequence of simple row or column stretchings but not both, and if the glue is chosen by Theorem 5, and if the stretched matrix is computed in finite precision arithmetic with unit roundoff  $\epsilon$ , then the computed  $\overline{A^S}$  differs from the ideal  $A^S$  as follows*

$$\overline{A^S} = A^S + E \quad \|E\|_p \leq c_{5.1} [(1 + \epsilon)^n - 1] \|A\|_p$$

where  $n$  is the order of  $A$  and  $c_{5,1}$  is given by the table

$c_{5,1}$	$p = 1$	$p = \infty$
row	2	$m$
column	$m$	2

in which each row of  $A$  stretches to at most  $m$  rows of  $A^S$ . When the sequence of stretchings is disjoint in the sense that later stretchings do not alter the rows or columns of earlier stretchings, then 2 can replace  $m$  in this table.

*Proof for  $p = \infty$ .* In the column case, Definition 2's stretchings copy the entries of  $A$  to  $A^S$  and additionally place two pieces of Theorem 5's glue,  $\pm\|A\|_\infty/2$ , in new rows. The computed and ideal stretched matrices thus differ by a matrix  $E$  whose nonzero entries equal the error in computing  $\pm\|A\|_\infty/2$ . If  $\delta_c$  bounds this error then  $\|E\|_\infty \leq 2\delta_c$ .

In the row case, Definition 2's stretchings again copy the entries of  $A$  to  $A^S$  and place one or two pieces of Theorem 5's glue,  $\pm\|A\|_\infty$ , in the stretched rows. Should the stretchings in the sequence not alter rows stretched earlier, then each row acquires at most two pieces of glue, otherwise, a row could acquire as many as  $m - 1$ . If  $\delta_r$  bounds the error in computing  $\|A\|_\infty$  then  $\|E\|_\infty \leq (2 \text{ or } m - 1)\delta_r$ .

The standard model of machine computation gives to each arithmetic operation a small relative perturbation bounded by the optimistically named *unit roundoff*. The  $j$ -th absolute row sum of  $A$  may be computed as follows

$$\begin{aligned} s_{j,1} &= |a_{j,1}| & \bar{s}_{j,1} &= |a_{j,1}| \\ s_{j,2} &= s_{j,1} + |a_{j,2}| & \bar{s}_{j,2} &= (\bar{s}_{j,1} + |a_{j,2}|)(1 + \epsilon_2) \\ &\vdots & &\vdots \\ s_{j,n} &= s_{j,n-1} + |a_{j,n}| & \bar{s}_{j,n} &= (\bar{s}_{j,n-1} + |a_{j,n}|)(1 + \epsilon_n) \end{aligned}$$

in which bars denote computed quantities, sign changes are errorless, the  $a_{j,k}$  are entries of  $A$ , and  $|\epsilon_i| \leq \epsilon$ . The ideal and computed sums therefore are

$$s_{j,n} = \sum_{k=1}^n |a_{j,k}| \quad \bar{s}_{j,n} = \sum_{k=1}^n \left[ |a_{j,k}| \prod_{i=k \wedge 2}^n (1 + \epsilon_i) \right]$$

so the computed sum lies between  $(1 \pm \epsilon)^{n-1}$  of the ideal. The ideal and computed  $\|A\|_\infty$  are the largest  $s_{i,n}$  and  $\bar{s}_{j,n}$ , respectively. No computed sum exceeds  $(1 + \epsilon)^{n-1}$  of the largest ideal sum, so the computed  $\|A\|_\infty$  lies between  $(1 \pm \epsilon)^{n-1}\|A\|_\infty$ . Thus  $\delta_r \leq [(1 \pm \epsilon)^{n-1} - 1]\|A\|_\infty$  whence

$$\|E\|_\infty \leq (2 \text{ or } m - 1)[(1 \pm \epsilon)^{n-1} - 1]\|A\|_\infty$$

for row stretching. The weaker bound  $(2 \text{ or } m)[(1 \pm \epsilon)^n - 1]\|A\|_\infty$  is more concise. As for  $\delta_c$ , halving the computed  $\|A\|_\infty$  introduces one more relative perturbation with the result that  $\delta_c \leq [(1 \pm \epsilon)^n - 1]\|A\|_\infty/2$  hence

$$\|E\|_\infty \leq [(1 \pm \epsilon)^n - 1]\|A\|_\infty$$

for column stretching. *End of proof.*

**Lemma 2 to Theorem 6.** *If  $A \rightarrow A^S$  is a row or column stretching of a nonsingular matrix obtained from a sequence of simple row or column stretchings but not both, and if in the row case the glue is chosen by Theorem 5 (the column case may choose any glue), and if the vector operations used to solve linear equations are scatter  $y \rightarrow y^S$  and gather  $z \rightarrow z_S$  operations, and if  $\bar{z}$  is an approximate solution to the stretched equations  $A^S z = y^S$ , then  $\bar{x} := \bar{z}_S$  can be regarded as an approximate solution to the unstretched equations  $Ax = y$ . The error in  $\bar{z}$  bounds the error in  $\bar{x}$*

$$\frac{\|x - \bar{x}\|_p}{\|x\|_p} \leq c_{5.2} \frac{\|z - \bar{z}\|_p}{\|z\|_p}$$

where  $c_{5.2}$  is given by the table

$c_{5.2}$	$p = 1$	$p = \infty$
row	$2m - 1$	1
column	$m$	1

in which  $n$  is the order of  $A$  and each row of  $A$  stretches to at most  $m$  rows of  $A^S$ .

*Proof.* When the squeezing  $z \rightarrow z_S := {}^-Xz$  is a gather operation then  ${}^-X$  is a matrix of 0's and 1's with one 1 per row and no more than one per column.

$$\|x - \bar{x}\| = \|{}^-X(z - \bar{z})\| \leq \|{}^-X\| \|z - \bar{z}\| = \|z - \bar{z}\|$$

This and  $\|z\| \leq c_{5.2}\|x\|$  will imply the Lemma's bound on the relative error.

For a column stretching,  $z = Xx$  (by the Corollary to Theorem 2), in which  $X$  is a matrix of 0's and 1's with exactly one 1 per row and from one to  $m$  per column (parts 2 and 6 for the column case of Lemma 1 to Theorem 5). Hence  $\|X\| = c_{5.2}$  and  $\|z\| \leq c_{5.2}\|x\|$ .

For a row stretching,  $Y^-$  can be any right inverse of the matrix  $Y$  that parameterizes the stretching in Definition 1

$$A^S = [B \quad G] P^t \quad YB = A \quad YG = 0$$

but when  $y \rightarrow y^S = Y^-y$  is a scatter operation then

$$Y^- = Q \begin{bmatrix} I \\ 0 \end{bmatrix}$$

in which  $I$  is an identity matrix and  $Q$  is a permutation matrix. Let

$$Y = [I \quad Y_2] Q^t \quad B = Q \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \quad G = Q \begin{bmatrix} G_1 \\ G_2 \end{bmatrix}$$

match the blocking of  $Y^-$ . Lemma 4 to Theorem 5 shows

$$(A^S)^{-1} = P \begin{bmatrix} A^{-1} & A^{-1}Y_2 \\ -G_2^{-1}B_2A^{-1} & G_2^{-1}(I - B_2A^{-1}Y_2) \end{bmatrix} Q^t$$

which provides a formula for  $z$ .

$$z = (A^S)^{-1}y^S = (A^S)^{-1}Y^-y = (A^S)^{-1}Y^-Ax = P \begin{bmatrix} I \\ -G_2^{-1}B_2 \end{bmatrix} x$$

With Theorem 5's choice of glue the Lemma also provides the following bounds.

$$\sigma = \|A\|_1/2 \implies \|G_2^{-1}B_2\|_1 \leq \|A\|_1(m-1)/\sigma = 2(m-1)$$

$$\sigma = \|A\|_\infty \implies \|G_2^{-1}B_2\|_\infty \leq \|A\|_\infty/\sigma = 1$$

Thus,  $c_{5.2}$  is  $2m - 1$  for the 1-norm, and 1 for the  $\infty$ -norm. *End of proof.*

**Theorem 6.** *If  $A \rightarrow A^S$  is stretching of a nonsingular matrix obtained from a sequence of simple row or column stretchings but not both, and if glue is chosen by Theorem 5, and if the stretched matrix  $\overline{A^S}$  is computed in finite precision arithmetic with unit roundoff  $\epsilon$ , and if the vector operations used to solve linear equations are error-free scatter  $y \rightarrow y^S$  and gather  $z \rightarrow z_S$  operations, and if the approximate solution  $\bar{z}$  to the computed stretched equations  $\overline{A^S}z = y^S$  exactly satisfies some perturbed equations  $(\overline{A^S} + E)\bar{z} = y^S$ , then*

$$\delta_1 := c_1 [(1 + \epsilon)^n - 1] < 1 \quad \text{and} \quad \delta_2 := \frac{\|E\|_p}{\|\overline{A^S}\|_p} < \frac{1 - \delta_1}{1 + \delta_1}$$

imply

$$\begin{aligned} \frac{\|x - \bar{z}_S\|_p}{\|x\|_p} &< c_2 \kappa(A) \frac{(\delta_1 + \delta_2 + \delta_1 \delta_2)}{1 - (\delta_1 + \delta_2 + \delta_1 \delta_2)} \\ &\approx c_2 \kappa(A) \left( c_1 n \epsilon + \|E\|_p / \|\overline{A^S}\|_p \right) \end{aligned}$$

in which  $c_1$  and  $c_2$  are given by the tables

$c_1$	$p = 1$	$p = \infty$	$c_2$	$p = 1$	$p = \infty$
row	2	$m$	row	$(2m - 1)^2$	$m^2$
column	$m$	2	column	$m^3$	$2m - 1$

where  $n$  is the order of  $A$  and each row of  $A$  stretches to at most  $m$  rows of  $A^S$ . When the sequence of stretchings is disjoint in that later stretchings do not alter the rows or columns of earlier stretchings, then the tables can be replaced by the ones below.

$c_1$	$p = 1$	$p = \infty$	$c_2$	$p = 1$	$p = \infty$
row	2	2	row	$(2m - 1)^2$	$3m$
column	2	2	column	$3m^2$	$2m - 1$

Thus, if  $A$  is well-conditioned, if  $\epsilon$  and  $\|E\|_p / \|\overline{A^S}\|_p$  are very small, and if  $m$  and  $n$  are not excessively large, then  $\bar{z}_S$  is a good approximate solution to  $Ax = y$ .

*Proof.* Lemma 1 shows  $\overline{A^S} = A^S + E_{5.1}$  with  $\|E_{5.1}\| \leq \delta_1 \|A\|$  hence  $\|E_{5.1}\| \leq \delta_1 \|A^S\|$  because Theorem 5's glue makes  $\|A\| \leq \|A^S\|$ . Therefore

$$\begin{aligned} \|E_{5.1} + E\| &\leq \|E_{5.1}\| + \|E\| \\ &= \|E_{5.1}\| + \delta_2 \|\overline{A^S}\| \\ &\leq \delta_1 \|A^S\| + \delta_2 (1 + \delta_1) \|A^S\| \\ &= (\delta_1 + \delta_2 + \delta_1 \delta_2) \|A^S\| \end{aligned}$$

in which the hypotheses for  $\delta_1$  and  $\delta_2$  assure  $\delta_1 + \delta_2 + \delta_1 \delta_2 < 1$ .<sup>5</sup> From

$$(A^S + E_{5.1} + E)\bar{z} = (\overline{A^S} + E)\bar{z} = y^S$$

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<sup>5</sup> The bound on  $\delta_1$  also implies  $\overline{A^S}$  is nonsingular, but this fact is not needed.

the matrix perturbation inequality of Section 4 now implies

$$\frac{\|z - \bar{z}\|}{\|z\|} \leq \kappa(A^S) \frac{\|E_{5.1} + E\|}{\|A^S\| - \|E_{5.1} + E\|} < \kappa(A^S) \frac{(\delta_1 + \delta_2 + \delta_1\delta_2)}{1 - (\delta_1 + \delta_2 + \delta_1\delta_2)}$$

while Lemma 2 and Theorem 5

$$\frac{\|x - \bar{x}\|}{\|x\|} \leq c_{5.2} \frac{\|z - \bar{z}\|}{\|z\|} \quad \text{and} \quad \kappa_p(A^S) \leq c_4 \kappa_p(A)$$

complete the chain of inequalities with  $c_2 = c_{5.2}c_4$ . *End of proof.*

**Theorem 7.** *An  $n \times n$ , dense system of linear equations can be solved by triangular factorization with row reordering for stability using*

$$\begin{aligned} 2n^3/3 - 2n/3 & \text{ arithmetic operations for the factorization and} \\ 2n^2 - n & \text{ operations for the solution phase.} \end{aligned}$$

*However, if the matrix is banded with strict lower and upper bandwidths  $\ell$  and  $u$ , and if  $\ell + u < n$ , then the operations reduce to*

$$\begin{aligned} 2\ell(\ell + u + 1)n - \ell(4\ell^2 + 6\ell u + 3u^2 + 6\ell + 3u + 2)/3 & \text{ for the factorization and} \\ (4\ell + 2u + 1)n - (2\ell^2 + 2\ell u + u^2 + 2\ell + u) & \text{ for the solution phase.} \end{aligned}$$

*Proof.* The factorization algorithm proceeds down the main diagonal of the coefficient matrix  $A$  by subtracting multiples of the row in which the diagonal entry lies from lower rows to place zeroes in the column beneath the diagonal entry. This column is first inspected and the rows reordered to insure that the diagonal entry has larger magnitude than any below. At the  $k^{th}$  diagonal entry of the dense matrix,  $n - k$  comparisons select the largest entry,  $n - k$  divisions form the multipliers, and  $(n - k)^2$  each of multiplications and subtractions perform the row operations. The total of these over all diagonal entries is

$$\sum_{k=1}^n [2(n - k) + 2(n - k)^2] = 2n^3/3 - 2n/3.$$

There results a factorization of the reordered matrix,  $PA = LU$ , in which  $P$  is the permutation matrix of the row reordering,  $L$  is the unit lower triangular matrix of multipliers, and  $U$  is the upper triangular matrix that contains what remains of the original matrix. A particular system of linear equations  $Ax = y$  can then be written as  $P^t L U x = y$  and solved by substitution. Substitution with  $L$  performs one multiplication and one subtraction for each entry below the main diagonal. Substitution with  $U$  additionally divides by each diagonal entry. Altogether

$$(n - 1)n + n + (n - 1)n = 2n^2 - n$$

operations are needed to obtain the solution.

The lower and upper triangular factors of banded matrices inherit the lower and upper bandwidths of the original. The effect of row reordering is merely to increase the upper bandwidth by the lower bandwidth [7]. The reordering performs one comparison, and the preparation of multipliers performs one division, for each lower diagonal entry. These account for

$$\sum_{j=1}^{\ell} 2(n - j) = 2\ell n - (\ell^2 + \ell).$$

operations. The  $j$  in this summation indexes the lower diagonals. The row operations subtract multiples of each strictly upper triangular entry from the  $\ell$  entries immediately below. With  $\ell + u < n$  upper diagonals, the first  $n - \ell$  rows account for

$$2\ell[(\ell + u)(n - \ell) - u(u + 1)/2] = 2\ell(\ell + u)n - \ell(2\ell^2 + 2\ell u + u^2 + u)$$

operations, and the final  $\ell$  rows account for

$$\sum_{k=1}^{\ell} 2(\ell - k)^2 = \ell(2\ell^2 - 3\ell + 1)/3.$$

The sum of the three expressions above simplifies to the Theorem's formula for the factorization phase. By reasoning similar to the dense case, the substitution phase of the banded case performs

$$2 \left[ \ell n - \frac{\ell(\ell + 1)}{2} \right] + n + 2 \left[ (\ell + u)n - \frac{(\ell + u)(\ell + u + 1)}{2} \right]$$

operations. *End of proof.*

**Theorem 8.** *This row and column partitioning makes a banded matrix into a block-bidiagonal one. For a matrix of order  $n$  with strict lower and upper bandwidths  $\ell$  and  $u$ , and with  $0 < \ell + u < n$ , the columns and rows partition into blocks of the following size.*

$$\begin{array}{ll} \text{columns} & a + u, \quad \ell + u, \quad \dots, \quad \ell + u, \quad \ell + c \\ \text{rows} & a, \quad u + \ell, \quad u + \ell, \quad \dots, \quad u + \ell, \quad c \\ & 0 \leq a \leq \ell \quad 0 \leq c \leq u \quad 0 < a + c \end{array}$$

The block-column dimension is  $m = \lceil n/(\ell + u) \rceil$ , and the block-row dimension is  $m + 1$  or  $m$  (since one of  $a$  or  $c$  may be zero). Moreover, the upper diagonal blocks are lower triangular and the lower diagonal blocks are upper triangular.

*Proof.* Since  $n$  is strictly greater than  $\ell + u$  it can be decomposed as

$$n = \left\lceil \frac{n - \ell - u}{\ell + u} \right\rceil (\ell + u) + r$$

where  $0 < r \leq \ell + u$ . Thus, any of the sums  $r = a + c$  with  $0 \leq a \leq \ell$  and  $0 \leq c \leq u$  produce the claimed partitioning of the columns by way of the decomposition

$$n = (a + u) + (m - 2)(\ell + u) + (\ell + c)$$

in which

$$m = \left\lceil \frac{n - \ell - u}{\ell + u} \right\rceil + 1 = \left\lceil \frac{n}{\ell + u} \right\rceil.$$

There are exactly  $m$  blocks of columns because neither  $a + u$  nor  $\ell + c$  can be zero, for example,  $0 < r = a + c \leq a + u$ . The row partitioning stems similarly from the decomposition

$$n = a + (m - 1)(\ell + u) + c$$

with the precise number of blocks varying from  $m + 1$  to  $m$  because one of  $a$  and  $c$  can be zero.

The matrix can be enlarged by placing  $\ell + u - a$  rows of zeroes at the top and  $\ell + u - c$  at the bottom,  $\ell - a$  columns of zeroes at the left and  $u - c$  at the right. The augmented matrix has block dimension  $(m + 1) \times m$  in which every block is square of order  $\ell + u$ . Entry  $(i, j)$  of the original matrix becomes entry  $(\tilde{i}, \tilde{j}) = (\ell + u - a + i, \ell - a + j)$  of the larger matrix. If the entry is nonzero, then  $-\ell \leq j - i \leq u$  because the original matrix has lower and upper bandwidths  $\ell$  and  $u$ , and then by a little arithmetic  $-(\ell + u) \leq \tilde{j} - \tilde{i} \leq 0$ . Decomposing

$$\begin{aligned}\tilde{i} &= \lfloor \tilde{i}/(\ell + u) \rfloor (\ell + u) + b \\ \tilde{j} &= \lfloor \tilde{j}/(\ell + u) \rfloor (\ell + u) + d\end{aligned}$$

in which  $0 \leq (b \text{ and } d) < \ell + u$ , it then follows that

$$-2(\ell + u) < -(\ell + u) + d - b \leq \tilde{j} - \tilde{i} + d - b = (\lfloor \tilde{j}/(\ell + u) \rfloor + \lfloor \tilde{i}/(\ell + u) \rfloor) (\ell + u).$$

Thus  $-1 \leq \lfloor \tilde{j}/(\ell + u) \rfloor - \lfloor \tilde{i}/(\ell + u) \rfloor \leq 0$ , and since  $\lfloor \tilde{i}/(\ell + u) \rfloor$  and  $\lfloor \tilde{j}/(\ell + u) \rfloor$  are the block indices of the nonzero entry, the inequality above means the entry lies in either a diagonal block or a block immediately below a diagonal block. The matrix therefore is block bidiagonal. The earlier inequality  $-(\ell + u) \leq \tilde{j} - \tilde{i} \leq 0$  means the nonzero region extends from the main diagonal of the subdiagonal blocks up to the main diagonal of the main diagonal blocks. The subdiagonal blocks therefore are upper triangular, and the main diagonal blocks are lower triangular. *End of proof.*

**Theorem 9.** *An order  $n + d$ , bordered, banded system of linear equations, whose coefficient matrix has  $d$  dense rows and columns in the bordering portion and has strict lower and upper bandwidths  $\ell$  and  $u$  in the  $n \times n$  banded portion, where  $0 < \ell + u < n$ , can be solved by simple row stretching and triangular factorization with row reordering for stability in*

$$\begin{aligned}& (4d^2 + 6d\ell + 2du + 2\ell^2 + 2\ell u + 2d + 2\ell)N \\ & - (d + \ell)(13d^2 + 14d\ell + 12du + 4\ell^2 + 6\ell u + 3u^2 + 9d + 6\ell + 3u + 2)/3\end{aligned}$$

*arithmetic operations for the factorization and*

$$\begin{aligned}& (4d + 4\ell + 2u + 1)N \\ & - (2d^2 + 4d\ell + 2du + 2\ell^2 + 2\ell u + u^2 + 2d + 2\ell + u)\end{aligned}$$

*operations for the solution phase, in which*

$$N = n + d \left\lceil \frac{n}{\ell + u} \right\rceil$$

*is the size of the stretched matrix.*

*Proof.* The simple row stretching of Definition 2, when based upon the partitioning of Theorem 8 and when accompanied by the reordering described in the text, results in a coefficient matrix with order  $N$ , with strict lower and upper bandwidths  $d + \ell$  and  $u$ , and with  $d$  dense columns but no dense rows. Without reordering, the triangular factors inherit this nonzero pattern. With row reordering, the upper bandwidth increases by the lower bandwidth as in the purely banded case because there are no dense rows. The results of Theorem 7 therefore account for everything

but the portion of the dense columns outside the increased band. Ignoring these, Theorem 7 reports the factorization phase performs

$$2(d + \ell)(d + \ell + u + 1)N \\ - (d + \ell) [4(d + \ell)^2 + 6(d + \ell)u + 3u^2 + 6(d + \ell) + 3u + 2] / 3$$

operations, and the solution phase performs

$$(4[d + \ell] + 2u + 1)N - [2(d + \ell)^2 + 2(d + \ell)u + u^2 + 2(d + \ell) + u].$$

The dense columns outside the increased band contain  $d(2N - 3d - 2\ell - 2u - 1)/2$  entries. The factorization phase subtracts multiples of these from the  $d + \ell$  entries below, for

$$d(d + \ell)(2N - 3d - 2\ell - 2u - 1)$$

more operations. The solution phase performs two operations for each strictly upper triangular entry, for

$$d(2N - 3d - 2\ell - 2u - 1)$$

more operations. The sums of the expressions above simplify to the formulas in the statement of the Theorem. *End of proof.*

## Appendix 2. Figure Explanations

This appendix explains the numerical experiments reported in the Figures. All calculations are performed by a Cray Y-MP8/264 with unit roundoff  $3.5 \times 10^{-15}$ . Matrix factorizations, solutions of linear equations, and singular values are computed using Linpack's SGEFA, SGESL and SSVDC [3].

**Figure 6.** *2-norm condition numbers for parameterized matrices of order 51 with sparsity patterns like the matrix in Figure 2.*

The parameterized matrices are tridiagonal except in their final rows and columns where all entries equal 1. The tridiagonal portion is a Toeplitz matrix with  $-1$  and  $-2$  on the lower and upper diagonals, and with the parameter on the main diagonal. There are 1201 matrices for parameter values uniformly distributed from  $-6$  to  $6$ . The condition numbers for parameter values between  $\pm 3$  exceed surrounding condition numbers by two orders of magnitude, but are still no worse than  $10^3$ . A parameter value near  $-3$  evidently produces a singular matrix, and nearby values produce matrices with high condition numbers—but not so high to trouble a machine with a  $3.5 \times 10^{-15}$  unit roundoff. The Figure's vertical axis has been lengthened to ease comparison with later figures.

**Figure 7.** *Maximum 2-norm relative errors for equations  $Ax = y$ , with 20 different  $y$ 's and the parameterized matrices  $A$  of Figure 6, solved by triangular factorization. The lower curve allows full row reordering. The upper curve restricts row reordering to the tridiagonal band.*

**Figure 8.** *Percent of non-zeroes in the triangular factors of the matrices of Figure 6. The upper curve allows full row reordering. The lower curve restricts row reordering to the tridiagonal band.*

Many right hand sides reduce the possibility of serendipity and smooth the curves by removing occasional outliers. In a sense, Figure 7 consists only of outliers because it reports the maximum error for any of the vectors, which repeat for each matrix. The vectors have entries uniformly and randomly distributed between  $\pm 1$ . With limited row reordering, the errors for parameter values between  $\pm 3$  exceed surrounding errors by a greater margin than do the condition numbers in Figure 6.

Restricted row reordering is performed by a modified SGEFA which limits its search for elimination rows as though the matrices were tridiagonal. Figure 8's percentages omit the main diagonals of the lower triangular factors, which are identically 1 and not stored. The downward spikes in Figure 8 indicate a few matrices have abnormally sparse factors.

**Figure 9.** *2-norm relative errors for the equations of Figure 7 solved by triangular factorization with full row reordering after stretching in the manner of Figure 3.*

**Figure 10.** *Percent of non-zeroes in the triangular factors of the stretched matrices of Figure 9. The percentages are relative to the size of the unstretched matrices.*

The Introduction cites Figures 9 and 10 but later sections more precisely explain the stretching. Section 3 defines simple row stretching. Section 5 applies the stretching to bordered, banded matrices. Theorem 9 shows the stretched matrices have size

$$N = n + d \left\lceil \frac{n}{\ell + u} \right\rceil = 50 + 1 \left\lceil \frac{50}{1 + 1} \right\rceil = 75$$

which is relatively much larger than the original matrices only because the bandwidth is very small in this example.

Figure 9 resembles Figure 7. The equations' right hand sides are those of the earlier Figure stretched by inserting zeroes as explained in Section 3. The relative errors are for the original variables, that is, they exclude the extraneous new variables also inserted by the stretching.

Figure 10 resembles Figure 8. Nonzeroes of the  $75 \times 75$  factors are reported as percentages of the  $51^2$  entries of the unstretched matrices. The percentages again omit the identically 1 main diagonal of one factor.

**Figure 12.** *Condition numbers of the matrices in Figure 6 (lower solid lines) and condition numbers after stretching (dashed lines) to remove either bordering rows or columns. Theorem 5 specifies glue that bounds (upper solid lines) either the 1- or  $\infty$ -norm condition numbers.*

The inverse matrices are explicitly formed to evaluate the 1-norm and  $\infty$ -norm condition numbers. The multiplier  $c$  in Theorem 5 is  $2m - 1$  or  $3m$ .

$$m = \left\lceil \frac{n}{\ell + u} \right\rceil = \left\lceil \frac{50}{1 + 1} \right\rceil = 25$$

**Figure 13.** *Smallest pivot and singular value for the banded portion of the matrices in Figure 6. Table 2's version of deflated block elimination assumes the pivot and singular value have the same magnitude "which is definitely not valid in general, but which is shown empirically and theoretically to be valid in practice" [2, p. 124].*

**Figure 14.** *Maximum 2-norm relative errors for equations  $Ax = y$  with 20 different  $y$ 's solved by block elimination, deflated block elimination, and simple row stretching. The parameterized coefficient matrices  $A$  are those of Figure 6. The elimination methods cannot be distinguished at this plotting resolution. The stretching data also appears in Figure 9.*

SGEFA's factorization is  $B = P(I + L)U$  in which  $P$  is a permutation matrix,  $L$  is strictly lower triangular and  $U$  is upper triangular. The smallest pivot is the entry  $u_{k,k}$  of smallest magnitude on the main diagonal of  $U$ . Chan [2] offers no guidance on the proper choice of the pivot's index. It appears to be  $k$ —disregarding the row permutation—because the deflated algorithm applies  $B^{-t} = P^t(I + L)^{-t}U^{-t}$  to column  $k$  of an identity matrix.

Figure 14 uses the same vectors  $y$  and reports the solution errors in the same manner as Figure 7. Block elimination and deflated block elimination are performed as shown in Tables 1 and 2.

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